Affine Sergeev Algebra and q-Analogues of the Young Symmetrizers for Projective Representations of the Symmetric Group

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We study a q-deformation for the semi-direct product of the symmetric group S_n with the Clifford algebra on n anticommuting generators. We obtain a q-version of the projective analogue for the classical Young symmetrizer introduced by the second author. Our main tool is an analogue of the Hecke algebra of complex valued functions on the group GL_n over a p-adic field relative to the Iwahori subgroup.

1 Introduction

The classical representation theory of the symmetric group S_n is well known. The irreducible representations of S_n over the complex field $\mathbb{C}$ are labelled by partitions ω of the integer n and realised through several constructions [**6**]. One of the most useful realisations employs primitive idempotents in the group ring $\mathbb{C}\cdot S_n$ now known as Young symmetrizers [**20**]. These idempotents in $\mathbb{C}\cdot S_n$ correspond to the Young standard tableaux Ω with n entries [**11**]. Suppose that a tableau Ω has shape ω. Let S_Ω and S^*_Ω be the subgroups in S_n respectively preserving the rows and columns of Ω as sets. The *Young symmetrizer* corresponding to Ω is

$$\frac{n_\omega}{n!}\sum_{s\in S_\Omega}\sum_{t\in S^*_\Omega} s\,t\cdot\operatorname{sgn}(t) \tag{1.1}$$

where n_ω denotes the number of all standard tableaux with shape ω. We will write $\Omega(i,j)$ for the entry of the tableau Ω in the row i and column j.

The q-analogues of Young symmetrizers are also known; for instance, see [**4**]. Here the group ring $\mathbb{C}\cdot S_n$ is replaced by the *Hecke algebra* $H_n(q)$ of type A_{n-1}. This is the algebra over the field $\mathbb{C}(q)$ generated by the elements $T_1,\dots,T_{n-1}$ with the relations

$$\begin{aligned}(T_k-q)(T_k+q^{-1}) &= 0\,;\\ T_kT_{k+1}T_k &= T_{k+1}T_kT_{k+1}\,;\\ T_kT_l &= T_lT_k\,,\quad l\neq k,k+1\,.\end{aligned} \tag{1.2}$$

As usual, we will write $T_s=T_{k_1}\cdots T_{k_l}$ for any choice of reduced decomposition $s=s_{k_1}\cdots s_{k_l}$ in the standard generators $s_1,\dots,s_{n-1}$ of the symmetric group S_n. For each partition ω there are two distinguished standard tableaux of shape ω obtained by inserting the symbols $1,2,\dots,n$ into the Young diagram of shape ω by rows and columns. These are called the *row* and *column tableaux* and are denoted by Ω^r and Ω^c respectively. Consider the elements of the algebra $H_n(q)$

$$E=\sum_{s\in S_{\Omega^r}} T_s\cdot q^{\,l}\,,\qquad E^*=\sum_{s\in S^*_{\Omega^c}} T_s\cdot(-q)^{-l}$$

where l is the length of the reduced decomposition of s. Now choose the element $s\in S_n$ such that $s(\Omega^r(i,j))=\Omega^c(i,j)$. Up to a certain non-zero multiplier from $\mathbb{C}(q)$, the product $E_\omega=E\,T_s^{-1}E^*T_s\in H_n(q)$ is the q-analogue [**4**, Section 2] of the Young symmetrizer (1.1) corresponding to $\Omega=\Omega^r$. That q-analogue is a primitive idempotent in the algebra $H_n(q)$.

Another approach to these q-analogues has been proposed by Cherednik [**1**]; it employs the *affine Hecke algebra* $He_n(q)$. This is the algebra over $\mathbb{C}(q)$ generated by $T_1,\dots,T_{n-1}$ and the pairwise commuting invertible elements $Y_1,\dots,Y_n$ subject to the relations

$$T_kY_kT_k \;=\; Y_{k+1}\,; \qquad T_kY_l \;=\; Y_lT_k\,, \quad l\neq k,k+1$$

along with the relations (1.2). Let χ be a character of the subalgebra in $He_n(q)$ formed by all Laurent polynomials in $Y_1,\dots,Y_n$. This subalgebra is maximal commutative; see [**2**]. Denote by M_χ the representation of $He_n(q)$ induced from the character χ, it is called a *principal series representation* [**16**]. The vector space M_χ can be identified with the algebra $H_n(q)$ so that the generators $T_1,\dots,T_{n-1}$ act via left multiplication. The action of the generators $Y_1,\dots,Y_n$ in M_χ is determined by $Y_k\cdot 1=\chi(Y_k)$. Note that the group S_n acts on the characters χ in a natural way: $w\cdot\chi(Y_k)=\chi(Y_{w^{-1}(k)})$ for any $w\in S_n$. The element $E_\omega\in H_n(q)$ has appeared in [**1**, Section 3] in the following guise: choose w such that $s\,w=w_0$ is the element of the maximal length in S_n and determine the character χ by

$$w\cdot\chi(Y_k)\;=\;q^{\,2(j-i)},\qquad k=\Omega^r(i,j)\,.$$

Then the operator of right multiplication in $H_n(q)$ by the element $E_\omega T_w$ is an intertwining operator $M_{w\cdot\chi}\to M_\chi$ over the algebra $He_n(q)$, cf. [**13**, Theorem 6.3]. This approach gives an explicit formula for the element E_ω different from that given above, cf. [**13**, Theorem 5.6].

It was Schur [**17**] who discovered non-trivial central extensions of the symmetric group S_n. In other words, the group S_n admits projective representations which cannot be reduced to linear ones. The analogues of the Young symmetrizers (1.1) for these representations were constructed by the second author in [**13**] using the approach of [**1**]. In [**13**] the role of the group ring $\mathbb{C}\cdot S_n$ is taken by the crossed product G_n of the group S_n with the Clifford algebra over $\mathbb{C}$ on n anticommuting generators. The irreducible G_n-modules are parametrised by the partitions λ of n with pairwise distinct parts. Note that the algebra G_n has a natural $\mathbb{Z}_2$-grading and here the notion of $\mathbb{Z}_2$-graded irreducibility is used. These modules can be constructed from the projective representations of the group S_n, see for instance [**19**].

A q-analogue of the algebra G_n was introduced by Olshanski in [**15**] by generalizing the double commutant theorem from [**7**]. It is the $\mathbb{C}(q)$-algebra $G_n(q)$ generated by the Hecke algebra $H_n(q)$ and the Clifford algebra with the generators $C_1,\dots,C_n$ subject to the relations (2.1) and (2.2) below. The algebra $G_n(q)$ is $\mathbb{Z}_2$-graded so that $\deg\,T_k=0$ and $\deg\,C_l=1$ for all possible k and l. The aim of the present paper is to construct the q-analogues in $G_n(q)$ of the projective Young symmetrizers from [**13**]. This is done by extending the approach of [**1**].

In Section 2 we examine some basic properties of the algebra $G_n(q)$. In particular, we show that the algebra $G_n(q)$ is semisimple; see Proposition 2.2. In Section 2 we also develop the combinatorics of shifted tableaux for partitions of n with pairwise distinct parts.

In Section 3 we introduce the main underlying object of the present paper. This is the algebra $Se_n(q)$ over $\mathbb{C}(q)$ generated by $G_n(q)$ and the pairwise-commuting invertible elements $X_1,\dots,X_n$ subject to the relations (3.1) and (3.2). It is the analogue for $G_n(q)$ of the affine Hecke algebra $He_n(q)$. We will call it the affine Sergeev algebra in honour of the author of [**18**] who extended the double commutant theorem of Schur and Weyl from the group S_n to the algebra G_n. The subalgebra in $Se_n(q)$ formed by all Laurent polynomials in $X_1,\dots,X_n$ is maximal commutative; see Proposition 3.2. By studying intertwining operators between the principal series representations of $Se_n(q)$ relative to this subalgebra, we obtain for the algebra $G_n(q)$ an analogue of the element $E_\omega T_w\in H_n(q)$.

There exists a homomorphism $\iota:Se_n(q)\to G_n(q)$ which is identical on the subalgebra $G_n(q)\subset Se_n(q)$; see Proposition 3.5. The images $J_1,\dots,J_n$ of the generators $X_1,\dots,X_n$

relative to this homomorphism are called the Jucys-Murphy elements of the algebra $G_n(q)$; cf. [**1**] and [**9**]. These elements play a major role in the present article.

In Section 4 we introduce a certain extension $\mathbb{F}$ of the field $\mathbb{C}(q)$. This will be a splitting field for the semisimple algebra $G_n(q)$ over $\mathbb{C}(q)$. As usual let us write for any non-negative integer m

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}\,.$$

The field $\mathbb{F}$ is obtained by adjoining to $\mathbb{C}(q)$ a square root of $[m]_{q^2}$ for each $m = 2, \ldots, n$. We assign to each standard shifted tableau Λ with n entries an element ψ_Λ in the extended algebra $G_n(q) \otimes_{\mathbb{C}(q)} \mathbb{F} = G'_n(q)$. This construction employs the fusion procedure introduced by Cherednik [**1**]; see Theorem 4.4 of the present paper. Each element ψ_Λ is an eigenvector for the left multiplications by the Jucys-Murphy elements $J_1, \ldots, J_n$. Moreover, for any $1 \leq k_1 < \cdots < k_p \leq n$, we have the equality in the algebra $G'_n(q)$

$$J_k \cdot C_{k_1} \cdots C_{k_p}\, \psi_\Lambda \;=\; \Big(\, [m+1]_{q^2} - [m]_{q^2} \mp (q - q^{-1}) \sqrt{\,[m+1]_{q^2} [m]_{q^2}}\,\Big) \cdot C_{k_1} \cdots C_{k_p}\, \psi_\Lambda$$

where $m = j - i$ for $k = \Lambda(i, j)$ and the sign $\mp$ depends on whether the index k is contained in the set $\{k_1, \ldots, k_p\}$ or not.

Now let Λ run through the set of all standard shifted tableaux corresponding to a fixed partition λ of n with pairwise distinct parts; here the number of parts will be denoted by ℓ_λ. Let us denote the row and column tableaux by Λ^r and Λ^c, respectively. Then the elements $C_{k_1} \cdots C_{k_p}\, \psi_\Lambda$ form a basis in the left ideal $V_\lambda \subset G'_n(q)$ generated by the element ψ_{Λ^r}; see Theorem 6.2. The vector space V_λ over $\mathbb{F}$ becomes a $G'_n(q)$-module under left multiplication. This module is not irreducible, but splits into a direct sum of $2^{[\,\ell_\lambda/2\,]}$ copies of a certain irreducible $G'_n(q)$-module U_λ; see Corollary 6.4. Here we again mean the $\mathbb{Z}_2$-graded irreducibility. The modules U_λ are non-equivalent for distinct λ and form a complete set of irreducible $G'_n(q)$-modules; see Corollary 6.8. Moreover, they remain irreducible on passing to any extension of the field $\mathbb{F}$; see Theorem 6.7.

Define the element $w_{\Lambda^r} \in S_n$ by the equality $s_{\Lambda^r} w_{\Lambda^r} = w_0$ where $s_{\Lambda^r}(\Lambda^r(i,j)) = \Lambda^c(i,j)$. The product $\psi_{\Lambda^r} T^{-1}_{w_{\Lambda^r}} \in G'_n(q)$ is our analogue of the above described element $E_\omega \in H_n(q)$. Corollaries 5.2 and 5.8 justify this claim.

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2 The Algebra G_n and its q-Deformation

Let us recall some results on the algebra G_n and examine its q-deformation $G_n(q)$ introduced by Olshanski [**15**] where it was referred to as the *Hecke-Clifford superalgebra*. The algebra G_n is defined as the crossed product of the symmetric group S_n with the Clifford algebra over the field $\mathbb{C}$ on n generators; it has was studied by Sergeev [**18**]. Here the Clifford algebra is generated over $\mathbb{C}$ by elements $C_1, \ldots, C_n$ obeying the relations

$$C_k^2 \;=\; -1\;; \quad C_k C_l \;=\; -\,C_l C_k\,, \quad l \neq k\,. \tag{2.1}$$

The action of the symmetric group on this algebra is given by the natural permutations on the generators: $s \cdot C_k = C_{s(k)} \cdot s$ for all $s \in S_n$. The algebra G_n may be regarded [**8**, **19**] as the twisted group algebra for a certain central extension of the Weyl group of type B_n.

The algebra G_n has a natural $\mathbb{Z}_2$-grading: any element $s \in S_n$ has degree zero, while each of the Clifford generators $C_1, \ldots, C_n$ has degree one. We will use the notion of $\mathbb{Z}_2$-graded irreducibility: a module over the $\mathbb{Z}_2$-graded algebra G_n is *irreducible* if the even part of its supercommutant equals $\mathbb{C}$. Furthermore, if the supercommutant coincides with $\mathbb{C}$ then the module is called *absolutely irreducible*.

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n is *strict* if all its parts are distinct, that is $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. Following the notation introduced by Morris [**12**], we write $\lambda \succ n$. The irreducible modules of the $\mathbb{Z}_2$-graded algebra G_n are parametrised by the strict partitions λ of n, while the irreducible G_n-module labelled by λ is absolutely irreducible if and only if the number ℓ_λ of parts in λ is even [**18**]. These two facts can be restated in the classical sense as follows: the irreducible representations over $\mathbb{C}$ of the algebra G_n are labelled by the pairs $\bigl(\lambda, (\pm 1)^{\ell_\lambda}\bigr)$ where $\lambda \succ n$; see [**13**, Section 1] and [**19**, Section 9].

Let q be an indeterminate over $\mathbb{C}$. For any positive integer n, let $G_n(q)$ denote the associative algebra with identity generated over the field $\mathbb{C}(q)$ by elements $T_1, T_2, \ldots, T_{n-1}$ and $C_1, C_2, \ldots, C_n$ subject to the following relations. The generators $T_1, \ldots, T_{n-1}$ obey the Hecke algebra relations (1.2), while the generators $C_1, \ldots, C_n$ satisfy the Clifford algebra relations (2.1). Furthermore, there are the cross relations

$$
\begin{aligned}
T_k C_k &= C_{k+1} T_k\,; \\
T_k C_{k+1} &= C_k T_k - (q - q^{-1})(C_k - C_{k+1})\,; \\
T_k C_l &= C_l T_k\,, \quad l \neq k, k+1
\end{aligned}
\tag{2.2}
$$

for all possible k, l. The algebra G_n may be recovered as the degenerate case where $q = 1$.

We will denote the element $q - q^{-1}$ by ε. Using the first relation in (1.2), the inverse of the Hecke generator T_k is $T_k - \varepsilon$; it is then apparent that the first two relations in (2.2) are equivalent. The algebra $G_n(q)$ has a natural $\mathbb{Z}_2$-grading: namely, the generators $T_1, \ldots, T_{n-1}$ are specified to be even while the Clifford generators $C_1, \ldots, C_n$ still have degree one.

The Clifford algebra has a natural basis

$$
\mathcal{C} \;=\; \bigl\{\, C_{k_1} \cdots C_{k_p} \bigm| 1 \le k_1 < \cdots < k_p \le n \,\bigr\}\,. \tag{2.3}
$$

Given any permutation $s \in S_n$, let us take any reduced decomposition $s = s_{j_1} \cdots s_{j_r}$ in terms of the standard Coxeter generators $s_1, s_2, \ldots, s_{n-1}$. Then as usual define the element $T_s \in G_n(q)$ by $T_s = T_{j_1} \cdots T_{j_r}$. The second and third relations in (1.2) imply that this definition does not depend on the reduced decomposition for s.

By the defining relations (1.2), (2.1) and (2.2), the elements $T_s \cdot C = T_s \cdot C_{k_1} \cdots C_{k_p}$ where $s \in S_n$ and $C \in \mathcal{C}$ form a linear basis in the $\mathbb{C}(q)$-algebra $G_n(q)$. In particular, we have

Proposition 2.1. *The algebra $G_n(q)$ has dimension $2^n \cdot n!$ over $\mathbb{C}(q)$.*

The next result is obtained by using the standard technique of [**5**].

Proposition 2.2. *The algebra $G_n(q)$ over $\mathbb{C}(q)$ is semisimple.*

Proof. We will verify that the $\mathbb{C}(q)$-algebra $G_n(q)$ has a zero radical. Introduce the associative algebra $G^*_n(q)$ generated by the elements $T_1, \ldots, T_{n-1}$ and $C_1, \ldots, C_n$ over the ring $\mathbb{C}[\,q, q^{-1}]$ of Laurent polynomials in q. We will view it as an infinite-dimensional algebra over $\mathbb{C}$. Suppose there exists a non-zero element $R \in \mathrm{rad}\; G_n(q)$; we will bring this to a contradiction. Multiplying R by some element in $\mathbb{C}[q]$, we can assume that $R \in G^*_n(q)$. On division by a suitable power of $q - 1$, we can assume further that $R \notin (q-1) \cdot G^*_n(q)$.

Define the $\mathbb{C}$-algebra homomorphism $\varpi : G^*_n(q) \to G_n$ by $T_k \mapsto s_k, C_k \mapsto C_k, q \mapsto 1$. We have the equality $s\,C \cdot \varpi(R) = \varpi(T_s C R)$ for any $s \in S_n$ and $C \in \mathcal{C}$. Since $\mathrm{rad}\; G_n(q)$ is a

nilpotent left ideal in $G_n(q)$ then the element $X \cdot \varpi(R)$ is nilpotent for any $X \in G_n$. Thus the left ideal $G_n \cdot \varpi(R)$ is contained in the radical of G_n. But the algebra G_n is semisimple, hence $\varpi(R) = 0$. Thus $R \in \ker \varpi = (q-1) \cdot G_n^*(q)$. This is the contradiction. ■

We will prove later that the irreducible $G_n(q)$-modules are parametrized by the same strict partitions of n as the irreducible G_n-modules; see Theorem 6.7. We conclude this section by describing the combinatorial objects known as shifted tableaux; these are analogous to the classical Young tableaux. The standard reference for these analogues is [**11**, Section III.7].

Let us fix a strict partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of the integer n. Then the *shifted Young diagram of shape* λ is the array of n boxes into l rows with λ_i boxes in the i-th row, such that each row is shifted by one position to the right relative to the preceding row. A *shifted tableau of shape* λ is an array obtained by inserting the symbols $1, 2, \dots, n$ bijectively into the n boxes of the shifted Young diagram for λ. We denote an arbitrary shifted tableau by the symbol Λ. The set of all shifted tableaux with shape λ is denoted by $\mathcal{T}_\lambda$. The symmetric group S_n acts transitively on the tableaux $\Lambda \in \mathcal{T}_\lambda$ by permutations on their entries, that is

$$(s \cdot \Lambda)(i,j) \;=\; s(\Lambda(i,j)) \qquad \text{for all } s \in S_n, \Lambda \in \mathcal{T}_\lambda.$$

A shifted tableau is *standard* if the symbols increase along each row (from left to right) and down each column; the subset of all standard tableaux in $\mathcal{T}_\lambda$ is written as $\mathcal{S}_\lambda$.

There are two distinguished elements in $\mathcal{S}_\lambda$: the *row tableau* Λ^r obtained by inserting the symbols $1, 2, \dots, n$ consecutively by rows into the shifted diagram of λ, and the *column tableau* Λ^c in which the symbols $1, 2, \dots, n$ occur consecutively by columns. For example, the row and column tableaux for $\lambda = (4,3,1) \succ 8$ are given respectively by

$$\Lambda^r = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & \end{array} \qquad \text{and} \qquad \Lambda^c = \begin{array}{cccc} 1 & 2 & 4 & 7 \\ & 3 & 5 & 8 \\ & & 6 & \end{array}.$$

Our construction will involve certain sequences of the integers $1, 2, \dots, n$ derived from these two special tableaux. However, the following notation is introduced for an arbitrary standard tableau $\Lambda \in \mathcal{S}_\lambda$. Let us denote by (Λ) the sequence of integers obtained from the shifted tableau Λ by reading the symbols along the rows from left to right ordered from the top row to the bottom row. Similarly, let $(\Lambda)^*$ denote the sequence derived from Λ by reading the symbols down the columns taken from the left column to the right column. The special tableaux described above satisfy the obvious properties $(\Lambda^r) = (\Lambda^c)^* = (1, 2, \dots, n)$.

Now let $\Lambda \in \mathcal{S}_\lambda$ be fixed. For each index $k = 2, \dots, n$, we denote by $\mathcal{A}_k$ and $\mathcal{B}_k$ the subsequences of (Λ) consisting of all entries $j < k$ which occur respectively after and before k in this sequence. Similarly, let $\mathcal{A}_k^*$ and $\mathcal{B}_k^*$ denote the subsequences in $(\Lambda)^*$ consisting of the entries $j < k$ which occur respectively after and before k in this column sequence. Denote by a_k, b_k and a_k^*, b_k^* the lengths of the sequences $\mathcal{A}_k, \mathcal{B}_k$ and $\mathcal{A}_k^*, \mathcal{B}_k^*$ respectively.

There is a bijection between the set $\mathcal{T}_\lambda$ and the symmetric group S_n described in the following way: given any shifted tableau Λ of shape λ, we define the permutation $w_\Lambda \in S_n$ by

$$w_\Lambda \;=\; \begin{pmatrix} 1 & 2 & \cdots & n \\ p_n & p_{n-1} & \cdots & p_1 \end{pmatrix}$$

where $(p_1, \dots, p_n)$ is the column sequence $(\Lambda)^*$. This bijection preserves the action of the symmetric group: $w_{s \cdot \Lambda} = s\, w_\Lambda$ for all $s \in S_n$. We will also require a second bijection $\mathcal{T}_\lambda \to S_n$ described as follows: given any tableau $\Lambda \in \mathcal{T}_\lambda$ the permutation $s_\Lambda \in S_n$ is specified by the rule $s_\Lambda \cdot \Lambda = \Lambda^c$. That is, s_Λ is the unique element which maps Λ onto the column tableau. Let $w_0 \in S_n$ be the element of maximal length; we have $w_0(k) = n + 1 - k$ for each k.

Lemma 2.3. *The permutations w_Λ and s_Λ obey the property $s_\Lambda w_\Lambda = w_0$ for any $\Lambda \in \mathcal{T}_\lambda$.*

The next result [**13**, Lemma 2.4] gives reduced decompositions for w_Λ and s_Λ where $\Lambda \in \mathcal{S}_\lambda$.

Lemma 2.4. *Given any standard tableau $\Lambda \in \mathcal{S}_\lambda$, there are reduced decompositions*

$$w_\Lambda = \prod_{k=2,\dots,n}^{\rightarrow} \left(\prod_{p=1,\dots,b_k^*}^{\rightarrow} s_{k-p} \right), \qquad s_\Lambda = \prod_{k=2,\dots,n}^{\leftarrow} \left(\prod_{p=1,\dots,a_k^*}^{\leftarrow} s_{k-p} \right).$$

Combining these results yields a reduced decomposition for the element w_0. In particular, for $\Lambda = \Lambda^c$ we obtain the obvious reduced decomposition

$$w_0 = \prod_{k=2,\dots,n}^{\rightarrow} \left(\prod_{p=1,\dots,k-1}^{\rightarrow} s_{k-p} \right).$$

The arrow notation on a product indicates the orientation of its (non-commuting) factors.

3 The Affine Sergeev Algebra

In this section, we introduce a certain object underlying the representation theory of $G_n(q)$. This will be referred to as the affine Sergeev algebra and denoted by $Se_n(q)$; it is a q-analogue of the degenerate affine Sergeev algebra employed in [**13**]. We will consider representations of $Se_n(q)$ induced from one-dimensional representations of a certain maximal commutative subalgebra; these are called principal series representations, cf. [**16**].

The *affine Sergeev algebra* $Se_n(q)$ is the associative unital algebra generated over the field $\mathbb{C}(q)$ by $G_n(q)$ together with the pairwise-commuting invertible elements $X_1, X_2, \dots, X_n$ subject to the relations

$$\begin{aligned} T_k X_k &= X_{k+1} T_k - \varepsilon\, (X_{k+1} - C_k C_{k+1} X_k)\,; \\ T_k X_{k+1} &= X_k T_k + \varepsilon\, (1 + C_k C_{k+1})\, X_{k+1}\,; \\ T_k X_l &= X_l T_k\,, \quad l \neq k, k+1 \end{aligned} \tag{3.1}$$

for all possible k, l and also the relations

$$C_k X_k = X_k^{-1} C_k\,; \quad C_k X_l = X_l C_k\,, \quad l \neq k\,. \tag{3.2}$$

Note that the first and second relations in (3.1) can be deduced from the single relation

$$(T_k - \varepsilon\, C_k C_{k+1})\, X_k T_k = X_{k+1}. \tag{3.3}$$

The centre $\mathrm{Z}(Se_n(q))$ will be precisely described in Proposition 3.2. However, for present purposes, it is sufficient to know that the centre contains any symmetric polynomial in the elements $X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}$. This can be verified by following [**13**, Proposition 3.1]. In particular, the square of the product

$$\prod_{1 \le k < l \le n} \left(X_k + X_k^{-1} - X_l - X_l^{-1} \right) \tag{3.4}$$

belongs to the centre $\mathrm{Z}(Se_n(q))$. Let $Se_n^\circ(q)$ denote the localisation of the affine Sergeev algebra by this central element. Since each factor in the product (3.4) can be written as

$$X_k^{-1} \left(X_k X_l - 1 \right) \left(X_k X_l^{-1} - 1 \right),$$

then it follows that the localisation $Se^{\circ}_n(q)$ contains the elements

$$\frac{1}{X_kX_l-1}\,,\quad \frac{1}{X_kX_l^{-1}-1}\,;\qquad 1\le k<l\le n. \tag{3.5}$$

This fact enables us to introduce certain elements in $Se^{\circ}_n(q)$ playing a key role in the sequel.

We will write $\mathbb{C}(q)[X]$ for the subalgebra $\mathbb{C}(q)[X_1^{\pm1},X_2^{\pm1},\dots,X_n^{\pm1}]$ in $Se_n(q)$; that is, the space of Laurent polynomials in the affine generators $X_1,\dots,X_n$. Let $\mathbb{C}(q)(X)$ denote the subalgebra in the localisation $Se^{\circ}_n(q)$ generated over $\mathbb{C}(q)[X]$ by the elements (3.5). Now for each index $k=1,2,\dots,n-1$, define the element $\Phi_k\in Se^{\circ}_n(q)$ by

$$\Phi_k \;=\; T_k+\frac{\varepsilon}{X_kX_{k+1}^{-1}-1}+\frac{\varepsilon}{X_kX_{k+1}-1}\cdot C_kC_{k+1}. \tag{3.6}$$

It follows from the defining relations (3.1), (3.2) that these elements satisfy the properties

$$\begin{aligned} \Phi_k\cdot X_k &= X_{k+1}\cdot\Phi_k\,;\\ \Phi_k\cdot X_{k+1} &= X_k\cdot\Phi_k\,;\\ \Phi_k\cdot X_l &= X_l\cdot\Phi_k\,,\quad l\ne k,k+1\,; \end{aligned} \tag{3.7}$$

cf. [**10**]. The next result is also established using (3.1) and (3.2); cf. [**13**, Proposition 3.2].

Proposition 3.1. *The elements $\Phi_1,\dots,\Phi_{n-1}$ obey the following relations in $Se^{\circ}_n(q)$.*

$$\begin{aligned} \Phi_k^2 &= 1-\varepsilon^2\cdot\left(\frac{X_kX_{k+1}^{-1}}{(X_kX_{k+1}^{-1}-1)^2}+\frac{X_k^{-1}X_{k+1}^{-1}}{(X_k^{-1}X_{k+1}^{-1}-1)^2}\right);\\ \Phi_k\Phi_{k+1}\Phi_k &= \Phi_{k+1}\Phi_k\Phi_{k+1}\,;\\ \Phi_k\Phi_l &= \Phi_l\Phi_k\,,\quad |k-l|>1. \end{aligned}$$

Given a permutation $s\in S_n$ and a reduced decomposition $s=s_{k_p}\cdots\, s_{k_1}$ define the element $\Phi_s\in Se^{\circ}_n(q)$ by $\Phi_s=\Phi_{k_p}\cdots\,\Phi_{k_1}$. The second and third relations in Proposition 3.1 imply that this definition is independent of the reduced decomposition. The equalities in (3.7) demonstrate that the adjoint action of each element $\Phi_k\in Se^{\circ}_n(q)$ on $X_1,\dots,X_n$ coincides with the standard action of the basic permutation $s_k\in S_n$. Thus

$$\Phi_s\cdot X_k \;=\; X_{s(k)}\cdot\Phi_s\,,\qquad k=1,\dots,n \tag{3.8}$$

for any permutation $s\in S_n$. The family of elements $\{\,\Phi_s\in Se^{\circ}_n(q)\mid s\in S_n\,\}$ will be used throughout this paper. Let us now prove the following result.

Proposition 3.2. *a) The subalgebra $\mathbb{C}(q)[X]$ in $Se_n(q)$ is maximal commutative.*
b) The centre $\mathrm{Z}(Se_n(q))$ of the affine Sergeev algebra consists precisely of the symmetric polynomials in the elements $X_k+X_k^{-1}\,,\ k=1,\dots,n$.

Proof. a) We follow an approach by Cherednik [**2**, Section 1]. First, we remark that for any $s\in S_n$ the element $\Phi_s\in Se^{\circ}_n(q)$ has the form

$$\Phi_s \;=\; T_s+\sum_{w}\sum_{C\in\mathcal{C}} f_{w,C}\cdot T_w\,C$$

for some $f_{w,C}\in\mathbb{C}(q)(X)$. Here the first summation is over the permutations $w\in S_n$ with $\mathrm{length}(w)<\mathrm{length}(s)$. Thus the elements Φ_sC for all $s\in S_n\,,\ C\in\mathcal{C}$ are linearly independent over $\mathbb{C}(q)(X)$. In particular, there is a direct sum decomposition

$$Se^{\circ}_n(q) \;=\; \bigoplus_{s\in S_n,C\in\mathcal{C}}\Phi_sC\cdot\mathbb{C}(q)(X) \;=\; \bigoplus_{s\in S_n,C\in\mathcal{C}}\mathbb{C}(q)(X)\cdot\Phi_sC\,. \tag{3.9}$$

Now consider any element $Z \in Se_n(q)$ in the centralizer of $\mathbb{C}(q)[X]$ as an element in $Se_n^\circ(q)$. Decompose this element relative to (3.9) as

$$Z = \sum_{s\in S_n} \sum_{C\in\mathcal{C}} z_{s,C} \cdot \Phi_s C \ ; \quad z_{s,C} \in \mathbb{C}(q)(X).$$

Suppose that $z_{s,C} \neq 0$ for some pair $(s,C) \neq (1,1)$. Then it follows from (3.9) that there exists $P \in \mathbb{C}(q)[X]$ such that $[Z,P] \neq 0$: if C is non-trivial then take $P = X_j$ corresponding to any letter C_j appearing in the word C. If $C = 1$ and s is non-trivial, then by the property (3.8) we take any polynomial in $\mathbb{C}(q)[X]$ not invariant under the action of $s \in S_n$. Thus $Z \in \mathbb{C}(q)(X)$. However, by definition, $Z \in Se_n(q)$. Hence $Z \in \mathbb{C}(q)[X]$.

b) The centre $\mathrm{Z}(Se_n(q))$ is contained in the maximal commutative subalgebra $\mathbb{C}(q)[X]$; that is, the central elements are Laurent polynomials in $X_1,\dots,X_n$. We will prove that these polynomials have the stated property. Firstly, let us check that every symmetric polynomial in the variables $X_1 + X_1^{-1},\dots,X_n + X_n^{-1}$ is central in $Se_n(q)$ by showing that any such polynomial P commutes with each of its generators.
i) The polynomial P obviously commutes with the affine generators $X_1,\dots,X_n$.
ii) The defining relations (3.2) show that the element $X_k + X_k^{-1}$ commutes with the Clifford generator C_l for any indices k and l.
iii) Using the definition (3.6) and (i),(ii) above, the statement that P commutes with the generators $T_1,\dots,T_{n-1}$ is equivalent to commutation with the elements $\Phi_1,\dots,\Phi_{n-1}$. Since the polynomial P is symmetric, the latter fact follows from property (3.8).

Hence, any symmetric polynomial in $X_1 + X_1^{-1},\dots,X_n + X_n^{-1}$ is a central element in $Se_n(q)$. Conversely, the same reasoning as in (a) establishes that the centre $\mathrm{Z}(Se_n(q))$ consists precisely of these elements. ■

Now fix a character χ of the maximal commutative subalgebra $\mathbb{C}(q)[X] \subset Se_n(q)$ valued in the algebraic closure $\mathbb{K}$ of the field $\mathbb{C}(q)$. This character is uniquely specified by its values $\chi(X_1),\chi(X_2),\dots,\chi(X_n)$ where each $\chi(X_k)$ lies in the multiplicative group $\mathbb{K}^*$ of $\mathbb{K}$. The symmetric group S_n has a natural action on the character χ: for any $s \in S_n$, we have

$$s\cdot\chi(X_k) \ = \ \chi(X_{s^{-1}(k)}) \ , \qquad k = 1,\dots,n.$$

Consider the representation π_χ over $\mathbb{K}$ of the algebra $Se_n(q)$ induced from the character χ. The space M_χ of the representation π_χ can be identified with the finite-dimensional $\mathbb{K}$-algebra $\bar{G}_n(q) = G_n(q)\otimes_{\mathbb{C}(q)}\mathbb{K}$. The generators $T_1,\dots,T_{n-1}$ and $C_1,\dots,C_n$ act in the representation space M_χ via the usual left multiplication; while the action of the elements $X_1,\dots,X_n$ is determined through the defining relations (3.1) and (3.2) in $Se_n(q)$ by

$$X_k\cdot m \ = \ (X_k m)\cdot 1 \qquad \text{for all } m \in M_\chi \, .$$

Here the action of X_k on the identity vector is specified through χ, that is $X_k \cdot 1 = \chi(X_k)$.

The character χ is said to be *generic* if $\chi(X_k) \neq \chi(X_l)^{\pm 1}$ for all $k \neq l$. For a generic character χ the action of $Se_n(q)$ in M_χ extends to each element $\Phi_s \in Se_n^\circ(q)$. This extended action is also denoted by π_χ. In the next section, we will see that the elements $\Phi_{w_\Lambda} \in Se_n^\circ(q)$ with $\Lambda \in \mathcal{S}_\lambda$ have a well-defined action in M_χ for certain non-generic characters χ; see Theorem 4.4. For the remainder of this section, we assume that the character χ is generic.

Proposition 3.3. *Given any permutation $s \in S_n$ then the operator μ_s of right multiplication in $\bar{G}_n(q)$ by the element $\pi_\chi(\Phi_s)(1)$ is an intertwining operator $M_{s\cdot\chi} \to M_\chi$.*

Proof. The action of the generators $T_1,\dots,T_{n-1}$ and $C_1,\dots,C_n$ in the representation spaces $M_\chi = M_{s\cdot\chi} = G_n^{\text{-}}(q)$ is through left multiplication and commutes with the operator μ_s. It therefore remains for us to verify that the action of the elements $X_1,\dots,X_n$ commutes with μ_s. Since the vector $1\in G_n^{\text{-}}(q)$ is cyclic for these actions, then it is sufficient to demonstrate that the operators $\pi_\chi(X_k)\cdot\mu_s$ and $\mu_s\cdot\pi_{s\cdot\chi}(X_k)$ coincide on the identity vector for each $k=1,2,\dots,n$. This result is established using the property (3.8) of the element Φ_s:

$$\pi_\chi(X_k)\,(\pi_\chi(\Phi_s)(1)) \;=\; \pi_\chi(X_k\Phi_s)(1) \;=\; \pi_\chi(\Phi_s X_{s^{-1}(k)})(1) \;=\; \pi_\chi(\Phi_s)\,(\pi_{s\cdot\chi}(X_k)(1)) \quad \blacksquare$$

The proof of Proposition 3.3 shows that the element $\pi_\chi(\Phi_s)(1)\in M_\chi$ is an eigenvector for each of the operators $\pi_\chi(X_k)$; we will need the following result in Section 6.

Corollary 3.4. *Given any $s\in S_n$ and $k=1,\dots,n$, we have the equality*

$$\pi_\chi(X_k)\,(\pi_\chi(\Phi_s)(1)) \;=\; (s\cdot\chi)(X_k)\cdot\pi_\chi(\Phi_s)(1).$$

Next, we introduce q-analogues for the Jucys-Murphy elements considered in [**13**]. Using these elements, we will describe a homomorphism $\iota : Se_n(q)\to G_n(q)$. For each $k=1,2,\dots,n$, define the *Jucys-Murphy element* $J_k\in G_n(q)$ inductively by

$$J_k \;=\; \begin{cases} 1 & \text{for } k=1\,;\\ (T_{k-1}-\varepsilon\, C_{k-1}C_k)\,J_{k-1}T_{k-1} & \text{for } k=2,\dots,n. \end{cases} \tag{3.10}$$

Proposition 3.5. *A homomorphism $\iota : Se_n(q)\to G_n(q)$ which is identical on the subalgebra $G_n(q)\subset Se_n(q)$ is uniquely specified by $X_1\mapsto 1$. Then $X_k\mapsto J_k$ for each $k=1,2,\dots,n$.*

Proof. We will verify that the elements $J_1,\dots,J_n\in G_n(q)$ satisfy the same relations with the generators $T_1,\dots,T_{n-1}$ and $C_1,\dots,C_n$ as the elements $X_1,\dots,X_n\in Se_n(q)$ respectively. By the definition (3.10) we have $T_l\,J_k = J_k\,T_l$ and $C_l\,J_k = J_k\,C_l$ for any $l>k$. The first two relations in (3.1) are equivalent to the single relation (3.3). But again by the definition (3.10) we have $(T_k-\varepsilon\,C_kC_{k+1})\,J_kT_k \;=\; J_{k+1}$. It now suffices to verify the following relations:

$$T_l\,J_{k+1} = J_{k+1}\,T_l\,,\quad l<k\,;\qquad C_{k+1}\,J_{k+1} = J_{k+1}^{-1}\,C_{k+1}\,;$$
$$C_l\,J_{k+1} = J_{k+1}\,C_l\,,\quad J_l\,J_{k+1} = J_{k+1}\,J_l\,,\qquad l\le k\,.$$

We will verify all these relations by induction on k. The initial case $k=0$ is trivial.

The relation $T_lJ_{k+1} = J_{k+1}T_l$ with $l<k-1$ immediately follows from the inductive assumption. In the remaining case $l=k-1$ we have $k\ge 2$ and

$$J_{k+1} \;=\; (T_k-\varepsilon\,C_kC_{k+1})(T_{k-1}-\varepsilon\,C_{k-1}C_k)J_{k-1}T_{k-1}T_k\,.$$

The relations (1.2) and (2.1),(2.2) then provide the equality

$$T_{k-1}J_{k+1} \;=\; (T_k-\varepsilon\,C_kC_{k+1})(T_{k-1}-\varepsilon\,C_{k-1}C_k)T_kJ_{k-1}T_{k-1}T_k\,. \tag{3.11}$$

But the element T_k commutes with J_{k-1}. The right hand side of (3.11) then equals $J_{k+1}T_{k-1}$ by the second relation in (1.2).

By using the definition (3.10), the product $C_{k+1}\,J_{k+1}$ can be expressed as $(T_k-\varepsilon)\,C_k\,J_k\,T_k$. The inductive assumption provides the equality $C_kJ_k = J_k^{-1}C_k$. Using the relations (1.2) and (2.1),(2.2) again, the above expression becomes

$$(T_k-\varepsilon)\,J_k^{-1}\,(T_kC_{k+1}+\varepsilon\,(C_k-C_{k+1})) \;=\; T_k^{-1}J_k^{-1}\,(T_k-\varepsilon\,C_kC_{k+1})^{-1}\;C_{k+1} \;=\; J_{k+1}^{-1}\,C_{k+1}\,.$$

The equality $C_l\, J_{k+1} = J_{k+1}\, C_l$ with $l < k$ also follows immediately from the inductive assumption. Let us check the equality $C_k\, J_{k+1} = J_{k+1}\, C_k$. The product $C_k\, J_{k+1}$ can be written as $(T_k - \varepsilon\, C_k C_{k+1}) C_{k+1} J_k T_k$. But the factors J_k and C_{k+1} here commute, and the result follows by the first relation in (2.2).

Now using (3.10) along with the inductive assumption, we get $J_l\, J_{k+1} = J_{k+1}\, J_l$ for each $l < k$. The remaining case $l = k$ can be settled by writing the element J_k as the product $(T_{k-1} - \varepsilon\, C_{k-1} C_k) J_{k-1} T_{k-1}$. The element J_{k+1} commutes with each factor here. ■

Let us now determine the character χ by introducing an array with entries in $\mathbb{K}^*$. Fix an array of shifted shape λ with entries $x(i,j) \in \mathbb{K}^*$ and determine the values $\chi(X_1^{-1}), \dots, \chi(X_n^{-1})$ by

$$w_{\Lambda^r} \cdot \chi(X_k^{-1}) \;=\; x(i,j)\;, \qquad k = \Lambda^r(i,j).$$

In fact, we then have the equality

$$w_{\Lambda} \cdot \chi(X_k^{-1}) \;=\; x(i,j)\;, \qquad k = \Lambda(i,j) \tag{3.12}$$

for any shifted tableau $\Lambda \in \mathcal{T}_\lambda$. The reason for specifying the values $\chi(X_k^{-1})$ rather than the usual values $\chi(X_k)$ is purely notational and will become apparent later on.

Consider the action of the elements $\Phi_s \in Se_n^\circ(q)$ on the identity vector $1 \in M_\chi$ for any generic character χ. First, let us present some additional notation. For any $k = 1, 2, \dots, n-1$ let us introduce the rational function of $x, y \in \mathbb{K}$ valued in the algebra $G_n^{-}(q)$

$$\psi_k(x,y) \;=\; T_k + \frac{\varepsilon}{x^{-1}y - 1} + \frac{\varepsilon}{xy - 1}\, C_k C_{k+1}\,. \tag{3.13}$$

The action of $\Phi_1, \dots, \Phi_{n-1}$ on the identity $1 \in M_\chi$ is determined through the relations (3.1) and (3.2) : for $\chi(X_k^{-1}) = x$ and $\chi(X_{k+1}^{-1}) = y$, we have

$$\pi_\chi(\Phi_k)(1) \;=\; \psi_k(x,y)\,. \tag{3.14}$$

Let us fix a standard tableau $\Lambda \in \mathcal{S}_\lambda$. The final result in the present section generalises the equality (3.14) by describing how the element Φ_{w_Λ} acts in the representation M_χ. It follows directly from Lemma 2.3 and Proposition 3.3 that

$$\pi_\chi(\Phi_{w_0})(1) \;=\; \pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \cdot \pi_\chi(\Phi_{w_\Lambda})(1). \tag{3.15}$$

Using Lemma 2.4, we obtain the following decompositions for the factors on the right hand side in (3.15); see [**13**, Section 4] for a detailed proof of this result.

Proposition 3.6. *For each $k = 1, 2, \dots, n$ let us define $x_k = x(i,j)$ where $k = \Lambda(i,j)$. Then*

$$\pi_\chi(\Phi_{w_\Lambda})(1) \;=\; \overrightarrow{\prod_{k=2,\dots,n}} \left(\overrightarrow{\prod_{p=1,\dots,b_k^*}} \psi_{k-p}(x_k, x_{\mathcal{B}_k^*(p)}) \right),$$

$$\pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \;=\; \overleftarrow{\prod_{k=2,\dots,n}} \left(\overleftarrow{\prod_{q=1,\dots,a_k^*}} \psi_{k-q}(x_{\mathcal{A}_k^*(a_k^*-q+1)}, x_k) \right).$$

In the next section we study the elements $\pi_\chi(\Phi_{w_\Lambda})(1)$ for some non-generic characters χ.

4 The Elements ψ_Λ in the Algebra $G'_n(q)$

We start this section with introducing certain finite extension $\mathbb{F}$ of the field $\mathbb{C}(q)$. In Section 6 we will show that $\mathbb{F}$ is a splitting field for the semisimple algebra $G_n(q)$ over $\mathbb{C}(q)$. We write

$$[m]_{q^2} = \frac{q^{2m} - q^{-2m}}{q^2 - q^{-2}}$$

for any integer m. Notice that $[0]_{q^2} = 0$ and $[1]_{q^2} = 1$. The field $\mathbb{F}$ is obtained from $\mathbb{C}(q)$ by adjoining a square root of $[m]_{q^2}$ for each $m = 2, \dots, n$. The $\mathbb{F}$-algebra $G_n(q) \otimes_{\mathbb{C}(q)} \mathbb{F}$ will be denoted by $G'_n(q)$, it is semisimple due to Proposition 2.2. In this section we will define an element $\psi_\Lambda \in G'_n(q)$ for each standard tableau $\Lambda \in \mathcal{S}_\lambda$ as a certain specialization of $\pi_\chi(\Phi_{w_\Lambda})(1)$. The element $\psi_{\Lambda^r} \in G'_n(q)$ associated with the row tableau Λ^r has particular significance. In Section 6 it will provide a q-analogue of the symmetrizer constructed in [**13**].

Consider the rational functions $\psi_k(x,y)$ of $x, y \in \mathbb{K}$ defined by (3.13). These functions are valued in the algebra $G_n^-(q)$.

Lemma 4.1. *The functions $\psi_k(x,y)$ satisfy the equations*

$$\begin{aligned} \psi_k(y,x)\,\psi_k(x,y) &= 1 - \varepsilon^2 \cdot \left(\frac{x^{-1}y}{(x^{-1}y-1)^2} + \frac{xy}{(xy-1)^2} \right); \\ \psi_k(x,y)\,\psi_l(z,w) &= \psi_l(z,w)\,\psi_k(x,y)\,, \qquad |k-l| \geq 2\,; \\ \psi_k(x,y)\,\psi_{k+1}(z,y)\,\psi_k(z,x) &= \psi_{k+1}(z,x)\,\psi_k(z,y)\,\psi_{k+1}(x,y) \end{aligned}$$

for all possible k and l. Furthermore, we also have the equalities

$$\psi_k(x,y)^2 \;=\; -\varepsilon \cdot \frac{x+y}{x-y} \cdot \psi_k(x,y) \,+\, 1 \,-\, \varepsilon^2 \cdot \left(\frac{x^{-1}y}{(x^{-1}y-1)^2} + \frac{xy}{(xy-1)^2} \right)$$

and

$$C_k\,\psi_k(x,y) = \psi_k(x,y^{-1})\,C_{k+1}\,, \qquad C_{k+1}\,\psi_k(x,y) = \psi_k(x^{-1},y)\,C_k\,.$$

Proof. Let χ be a generic character of $\mathbb{C}(q)[X]$ such that $\chi(X_k^{-1}) = x$ and $\chi(X_{k+1}^{-1}) = y$. Then it follows from Proposition 3.3 and the equality (3.14) that

$$\pi_\chi(\Phi_k^2)(1) \;=\; \pi_{s_k \cdot \chi}(\Phi_k)(1) \cdot \pi_\chi(\Phi_k)(1) \;=\; \psi_k(y,x)\,\psi_k(x,y).$$

On the other hand, the action of the element Φ_k^2 on the identity vector $1 \in M_\chi$ may be evaluated explicitly using the first relation in Proposition 3.1 :

$$\pi_\chi(\Phi_k^2)(1) \;=\; 1 - \varepsilon^2 \cdot \left(\frac{x^{-1}y}{(x^{-1}y-1)^2} + \frac{xy}{(xy-1)^2} \right).$$

The second equation in Lemma 4.1 is an immediate consequence of the relations (1.2) to (2.2). Now let us take a generic character χ of $\mathbb{C}(q)[X]$ such that $\chi(X_k^{-1}) = z$, $\chi(X_{k+1}^{-1}) = x$ and $\chi(X_{k+2}^{-1}) = y$. The second relation in Proposition 3.1 gives the equality

$$\pi_\chi(\Phi_k \Phi_{k+1} \Phi_k)(1) \;=\; \pi_\chi(\Phi_{k+1} \Phi_k \Phi_{k+1})(1)\,;$$

evaluating the actions on each side using Proposition 3.3 gives

$$\begin{aligned} &\pi_{s_{k+1}s_k \cdot \chi}(\Phi_k)(1) \cdot \pi_{s_k \cdot \chi}(\Phi_{k+1})(1) \cdot \pi_\chi(\Phi_k)(1) \\ &\qquad = \pi_{s_k s_{k+1} \cdot \chi}(\Phi_{k+1})(1) \cdot \pi_{s_{k+1} \cdot \chi}(\Phi_k)(1) \cdot \pi_\chi(\Phi_{k+1})(1)\,. \end{aligned}$$

The equality (3.14) implies that this is precisely the third equation in Lemma 4.1. The last three equalities can be established by direct computation; here the details are omitted. ■

Note that the third relation in Lemma 4.1 is the *Yang-Baxter equation* with the spectral parameters x, y and z; cf. [**7**]. Now consider the following condition on the pair (x, y):

$$\frac{x^{-1}y}{(x^{-1}y-1)^2} + \frac{xy}{(xy-1)^2} = \frac{1}{\varepsilon^2}\,. \tag{4.1}$$

This constraint is a q-analogue to the condition (4.11) in [**13**]; it will be referred to as the *idempotency condition* on (x, y) in view of the following consequence of Lemma 4.1.

Corollary 4.2. *a) Suppose that the pair* (x, y) *satisfies* (4.1) *and* $y \neq x, x^{-1}$ *then*

$$\psi_k(x,y)^2 = -\varepsilon \cdot \frac{x+y}{x-y} \cdot \psi_k(x,y)\,.$$

b) Suppose that the pair (x, y) *does not satisfy* (4.1) *and* $y \neq x, x^{-1}$. *Then* $\psi_k(x, y)$ *is invertible and*

$$\psi_k(x,y)^{-1} = \left[1 - \varepsilon^2 \cdot \left(\frac{x^{-1}y}{(x^{-1}y-1)^2} + \frac{xy}{(xy-1)^2}\right)\right]^{-1} \psi_k(y,x)\,.$$

Before proceeding any further, let us examine the idempotency condition (4.1) in more detail. Consider new variables (u, v) related to (x, y) by

$$\frac{x+x^{-1}}{2} = \frac{q\,u^2 + q^{-1}\,u^{-2}}{q+q^{-1}}\,, \qquad \frac{y+y^{-1}}{2} = \frac{q\,v^2 + q^{-1}\,v^{-2}}{q+q^{-1}}\,. \tag{4.2}$$

Performing this substitution, the idempotency condition (4.1) takes the form

$$\left(q^2 - u^2v^{-2} - u^{-2}v^2 + q^{-2}\right)\left(q^2 - q^2u^2v^2 - q^{-2}u^{-2}v^{-2} + q^{-2}\right) = 0\,; \tag{4.3}$$

this can be factorised as

$$\left(qu^2 - q^{-1}v^2\right)\left(qu^{-2} - q^{-1}v^{-2}\right)\left(q^{-1}u^{-2} - q^{-1}v^2\right)\left(q^3u^2 - q^{-1}v^{-2}\right) = 0\,.$$

Thus the equation (4.3) has four solutions:

$$v^2 = q^2u^2\,, \quad v^2 = q^{-2}u^2\,, \quad v^2 = u^{-2}\,, \quad v^2 = q^{-4}u^{-2}\,.$$

Note that the last two solutions can be obtained from the initial two by the transformation $u \mapsto q^{-1}u^{-1}$. This transformation exchanges the quadratic factors in (4.3); whereas the first equality in (4.2) remains invariant. Hence each pair (x, y) satisfying the condition (4.1) is obtained via the substitution (4.2) from a pair (u, v) obeying

$$v^2 = q^{\pm 2}\,u^2. \tag{4.4}$$

Now consider the rational function in the variables x, y, z

$$\psi_k(x,y)\,\psi_{k+1}(z,y)\,\psi_k(z,x) \tag{4.5}$$

on the left hand side of the third equation in Lemma 4.1. This function is regular only when $y \neq x^{\pm 1}$, $z \neq y^{\pm 1}$, $z \neq x^{\pm 1}$. However, on restriction such that (x, y) satisfies the idempotency

condition, the function (4.5) has a value at $z=y$. To formulate this result we introduce some notation. For $k=1,\dots,n-2$, define the rational function

$$\theta_k(x,y) \;=\; \psi_k(x,y)\cdot T_{k+1}\cdot\psi_k(y,x) \;-\; \varepsilon^2\cdot\psi_k(x,y)\left[\frac{x^{-1}y}{(x^{-1}y-1)^2}-\frac{xy}{(xy-1)^2}\,C_kC_{k+1}\right.$$
$$\left.+\,\frac{1}{(xy-1)(x^{-1}y-1)}\,C_{k+1}C_{k+2}+\frac{1}{(xy-1)(x^{-1}y-1)}\,C_{k+2}C_k\right]$$

valued in the algebra $G_n^-(q)$. Let $\mathcal{I}$ be the subset in $\mathbb{K}^3$ consisting of all triples (x,y,z) such that the pair (x,y) satisfies (4.1) and $y\neq x,x^{-1}$. Then we have the following simple lemma.

Lemma 4.3. *The restriction of the rational function* (4.5) *to* $\mathcal{I}$ *is regular at* $z=y,y^{-1}$. *At* $z=y$ *this restriction coincides with* $\theta_k(x,y)$.

Proof. The function (4.5) will be written in a form where the singular component at $z=y,y^{-1}$ can be removed using the constraint (4.1). The product (4.5) can be expanded as

$$\psi_k(x,y)\cdot T_{k+1}\cdot\psi_k(z,x) \;+\; \varepsilon\cdot\psi_k(x,y)\cdot T_k\left(\frac{1}{z^{-1}y-1}+\frac{C_kC_{k+2}}{zy-1}\right)$$
$$+\;\varepsilon^2\cdot\psi_k(x,y)\cdot\left(\frac{1}{(z^{-1}y-1)}\frac{1}{(z^{-1}x-1)}\;+\;\frac{1}{(z^{-1}y-1)}\frac{1}{(zx-1)}\,C_kC_{k+1}\right.$$
$$\left.+\;\frac{1}{(z^{-1}x-1)}\frac{1}{(zy-1)}\,C_{k+1}C_{k+2}\;+\;\frac{1}{(zx-1)}\frac{1}{(zy-1)}\,C_kC_{k+2}\right).$$

On restriction to $\mathcal{I}$ the pair (x,y) satisfies (4.1) with $y\neq x,x^{-1}$. Then $\psi_k(x,y)\cdot\psi_k(y,x)=0$ and adding

$$-\,\varepsilon\cdot\psi_k(x,y)\,\psi_k(y,x)\cdot\left(\frac{1}{z^{-1}y-1}+\frac{C_kC_{k+2}}{zy-1}\right)$$

does not alter the value of (4.5). Hence the restriction of (4.5) to $\mathcal{I}$ coincides with

$$\psi_k(x,y)\cdot T_{k+1}\cdot\psi_k(z,x)\;-\;\varepsilon^2\cdot\psi_k(x,y)\left[\frac{x^{-1}z}{(x^{-1}y-1)(x^{-1}z-1)}\;-\;\frac{xz}{(xy-1)(xz-1)}\,C_kC_{k+1}\right.$$
$$\left.+\;\frac{1}{(xy-1)(x^{-1}z-1)}\,C_{k+1}C_{k+2}\;+\;\frac{1}{(xz-1)(x^{-1}y-1)}\,C_{k+2}C_k\right].$$

This function is manifestly regular at $z=y,y^{-1}$. Moreover, at $z=y$ it equals $\theta_k(x,y)$. ■

In the previous section, the character χ has been determined through (3.12) by a shifted array $\{\,x(i,j)\in\mathbb{K}^*\,\}$ of shape λ. Let us now fix the standard tableau $\Lambda\in\mathcal{S}_\lambda$ and, as in Proposition 3.6, write $x_k=x(i,j)$ for $k=\Lambda(i,j)$. We will denote by $\mathcal{X}$ the subset in $(\mathbb{K}^*)^n$ consisting of all n-tuples $(x_1,\dots,x_n)$ such that $x_k\neq x_l^{\pm1}$ for $k\neq l$. These n-tuples correspond to the generic characters χ. Furthermore, denote by $\mathcal{Y}$ the subset in $(\mathbb{K}^*)^n$ consisting of all $(x_1,\dots,x_n)$ satisfying the following two conditions: first, if k and l occupy different diagonals in Λ then $x_k\neq x_l^{\pm1}$; secondly, if these two diagonals are not neighbouring then (x_k,x_l) does not satisfy (4.1). Finally, denote by $\mathcal{F}$ the subset in $(\mathbb{K}^*)^n$ consisting of all $(x_1,\dots,x_n)$ such that for any adjacent entries k,l in the same row of Λ then the pair (x_k,x_l) satisfies the idempotency condition (4.1) with $x_k+x_l\neq 0$.

If $k=\Lambda(i,j)$ the difference $j-i$ is called the *content* of the box in the shifted diagram λ occupied by the symbol k in Λ. Let us fix the *special* point $(q_1,\dots,q_n)\in(\mathbb{K}^*)^n$ where

$$q_k\;=\;[j-i+1]_{q^2}-[j-i]_{q^2}-\varepsilon\sqrt{[j-i+1]_{q^2}[j-i]_{q^2}}\;,\qquad k=\Lambda(i,j)\,. \tag{4.6}$$

Observe that $(q_1,\dots,q_n)\notin\mathcal{X}$, while $(q_1,\dots,q_n)\in\mathcal{F}\cap\mathcal{Y}$. Here the membership of $\mathcal{F}$ follows directly from the definition (4.6); see (4.2) and (4.4). Meanwhile, for arbitrary entries $k=\Lambda(i,j)$ and $l=\Lambda(i',j')$, the definition (4.6) implies that

$$\left(q_k^{-1}q_l-1\right)\left(q_k-q_l^{-1}\right) \;=\; \frac{2}{q+q^{-1}}\cdot\left(q^{2(j'-i')+1}-q^{2(j-i)+1}+q^{-2(j'-i')-1}-q^{-2(j-i)-1}\right)$$

where $j-i$ and $j'-i'$ are non-negative integers. This equality shows that $q_l=q_k$ or $q_l=q_k^{-1}$ if and only if $j-i=j'-i'$, that is k and l occupy the same diagonal in Λ. Thus the special point $(q_1,\dots,q_n)$ lies within $\mathcal{Y}$.

Now for each $\Lambda\in\mathcal{S}_\lambda$ let us define

$$\psi_\Lambda(x_1,\dots,x_n) \;=\; \pi_\chi(\Phi_{w_\Lambda})(1)$$

for any generic character χ. We consider $\psi_\Lambda(x_1,\dots,x_n)$ as a rational function of the variables $x_1,\dots,x_n$ valued in the algebra $G_n^{-}(q)$. By Proposition 3.6, we have

$$\psi_\Lambda(x_1,\dots,x_n) \;=\; \prod_{k=2,\dots,n}^{\longrightarrow}\left(\prod_{p=1,\dots,b_k^*}^{\longrightarrow}\psi_{k-p}(x_k,x_{\mathcal{B}_k^*(p)})\right) \tag{4.7}$$

where the sequences $\mathcal{B}_k^*$ are as defined in Section 2. This rational function may have poles outside the set $\mathcal{X}$. However, we will establish that its restriction to $\mathcal{F}$ is regular in $\mathcal{F}\cap\mathcal{Y}$. The continuation of the function (4.7) to the point $(q_1,\dots,q_n)$ along $\mathcal{F}$ is called the *fusion procedure,* this notion has been introduced by Cherednik [**1**].

Theorem 4.4. *For any standard tableau $\Lambda\in\mathcal{S}_\lambda$, the restriction of $\psi_\Lambda(x_1,\dots,x_n)$ to $\mathcal{F}$ is regular in $\mathcal{F}\cap\mathcal{Y}$. This restriction does not vanish at the point $(q_1,\dots,q_n)$.*

Proof. We will follow the constructive proof from [**13**, Theorem 5.6]. Firstly, we remark that Lemma 2.4 gives the reduced decomposition

$$w_\Lambda \;=\; \prod_{k=2,\dots,n}^{\longrightarrow}\left(\prod_{p=1,\dots,b_k^*}^{\longrightarrow}s_{k-p}\right)$$

for the element $w_\Lambda\in S_n$. By expanding the product (4.7) using the definition of the functions $\psi_k(x,y)$, we obtain a sum with leading term

$$T_{w_\Lambda} \;=\; \prod_{k=2,\dots,n}^{\longrightarrow}\left(\prod_{p=1,\dots,b_k^*}^{\longrightarrow}T_{k-p}\right)$$

while the remaining terms involve elements T_s with permutations $s\in S_n$ of smaller length. Therefore if the restriction of $\psi_\Lambda(x_1,\dots,x_n)$ is regular at a point, its value must be non-zero.

Our consideration may be restricted to the case for $\Lambda=\Lambda^c$. Namely, using the equalities in (3.15) and Proposition 3.6, we have

$$\prod_{k=2,\dots,n}^{\longleftarrow}\left(\prod_{q=1,\dots,a_k^*}^{\longleftarrow}\psi_{k-q}(x_{\mathcal{A}_k^*(a_k^*-q+1)},x_k)\right)\cdot\psi_\Lambda(x_1,\dots,x_n) \;=\; \psi_{\Lambda^c}(x_1',\dots,x_n') \tag{4.8}$$

where we write $x_k'=x(i,j)$ if $k=\Lambda^c(i,j)$. Each factor $\psi_{k-q}(x_{\mathcal{A}_k^*(a_k^*-q+1)},x_k)$ in the product on the left hand side of (4.8) is regular in $\mathcal{Y}$ and has invertible values. Hence we can assume that $\Lambda=\Lambda^c$. Then x_k' coincides with x_k for each k.

The function $\psi_{\Lambda^c}(x_1, \ldots, x_n)$ can be written as the product

$$\psi_{\Lambda^c}(x_1, \ldots, x_n) \;=\; \theta_{\Lambda^c}(x_1, \ldots, x_n) \cdot \theta'_{\Lambda^c}(x_1, \ldots, x_n) \tag{4.9}$$

where we denote

$$\theta_{\Lambda^c}(x_1, \ldots, x_n) \;=\; \prod_{k=2,\ldots,n}^{\rightarrow} \left(\prod_{p=1,\ldots,b_k}^{\rightarrow} \psi_{k-p}(x_k, x_{\mathcal{B}_k(p)}) \right), \tag{4.10}$$

$$\theta'_{\Lambda^c}(x_1, \ldots, x_n) \;=\; \prod_{k=2,\ldots,n}^{\leftarrow} \left(\prod_{q=1,\ldots,a_k}^{\leftarrow} \psi_{n-k+q}(x_k, x_{\mathcal{A}_k(a_k-q+1)}) \right). \tag{4.11}$$

Here, the sequences $\mathcal{B}_k$ and $\mathcal{A}_k$ are defined for Λ^c as described in Section 2. Each factor in the product (4.11) is regular in $\mathcal{Y}$. The proof is complete once we verify that the restriction of the function $\theta_{\Lambda^c}(x_1, \ldots, x_n)$ to $\mathcal{F}$ is regular in $\mathcal{F} \cap \mathcal{Y}$. The product (4.10) may contain factors which become singular within $\mathcal{Y} \setminus \mathcal{X}$. Specifically, these are the factors corresponding to pairs (k, p) where the symbols k and $\mathcal{B}_k(p)$ stand on the same diagonal in the tableau Λ^c. Let us call any such pair *singular*. Given any singular pair (k, p), we note that the symbols $\mathcal{B}_k(p)$ and $\mathcal{B}_k(p+1)$ are adjacent within some row of Λ^c.

Let $\hat{\theta}_{\Lambda^c}(x_1, \ldots, x_n)$ denote the product obtained from (4.10) by inserting the expression

$$\varepsilon^{-1} \cdot \frac{x_{\mathcal{B}_k(p)} - x_{\mathcal{B}_k(p+1)}}{x_{\mathcal{B}_k(p)} + x_{\mathcal{B}_k(p+1)}} \cdot \psi_{k-p-1}(x_{\mathcal{B}_k(p+1)}, x_{\mathcal{B}_k(p)}) \tag{4.12}$$

before each factor $\psi_{k-p}(x_k, x_{\mathcal{B}_k(p)})$ with singular (k, p). On restriction to $(x_1, \ldots, x_n) \in \mathcal{F}$, the pair $(x_{\mathcal{B}_k(p+1)}, x_{\mathcal{B}_k(p)})$ satisfies the condition (4.1) with $x_{\mathcal{B}_k(p+1)} \neq x^{\pm 1}_{\mathcal{B}_k(p)}$. Then the product

$$\psi_{k-p-1}(x_{\mathcal{B}_k(p+1)}, x_{\mathcal{B}_k(p)})\, \psi_{k-p}(x_k, x_{\mathcal{B}_k(p)})\, \psi_{k-p-1}(x_k, x_{\mathcal{B}_k(p+1)})$$

is regular at $x_k = x_{\mathcal{B}_k(p)}$ by Lemma 4.3. Using Lemma 4.1, it can be shown that this procedure of inserting the normalised factors (4.12) into the product (4.10) does not alter its restriction to $\mathcal{F}$. On the other hand, the rational function $\hat{\theta}_{\Lambda^c}(x_1, \ldots, x_n)$ is regular in $\mathcal{F} \cap \mathcal{Y}$. ■

Observe that for any tableau $\Lambda \in \mathcal{S}_\lambda$ the coordinates $q_1, \ldots, q_n$ belong to the subfield $\mathbb{F} \subset \mathbb{K}$. Hence for each $\Lambda \in \mathcal{S}_\lambda$ we can define the element $\psi_\Lambda \in G'_n(q)$ as the value at $(q_1, \ldots, q_n)$ of the restriction to $\mathcal{F}$ of $\psi_\Lambda(x_1, \ldots, x_n)$. Note that the proof of Theorem 4.4 together with Lemma 4.3 provides an explicit formula for the element ψ_Λ.

Let us make another important observation. Fix an index $k \in \{1, \ldots, n\}$. Consider the point $(q_1, \ldots, q_n)' \in \mathbb{F}$ obtained from $(q_1, \ldots, q_n)$ by inverting the coordinate q_k. We have $(q_1, \ldots, q_n)' \in \mathcal{F} \cap \mathcal{Y}$. Again using Theorem 4.4 define the element $\psi'_\Lambda \in G'_n(q)$ as the value at $(q_1, \ldots, q_n)'$ of the restriction to $\mathcal{F}$ of $\psi_\Lambda(x_1, \ldots, x_n)$. By using the last two equalities in Lemma 4.1 along with the definition (4.7) we obtain the following proposition.

Proposition 4.5. *We have the equality $C_k\, \psi_\Lambda = \psi'_\Lambda\, C_{w_\Lambda^{-1}(k)}$ in the algebra $G'_n(q)$.*

If

$$x \;=\; [a+1]_{q^2} - [a]_{q^2} - \varepsilon \sqrt{\,[a+1]_{q^2}[a]_{q^2}}\;, \quad y \;=\; [b+1]_{q^2} - [b]_{q^2} - \varepsilon \sqrt{\,[b+1]_{q^2}[b]_{q^2}}$$

for some non-negative integers $a \neq b$ then at $q \to 1$ the element $\psi_k(x, y) \in G'_n(q)$ degenerates to

$$s_k + \left(\sqrt{a(a+1)} - \sqrt{b(b+1)}\; \right)^{-1} - \left(\sqrt{a(a+1)} + \sqrt{b(b+1)}\; \right)^{-1} C_k C_{k+1} \in G_n$$

by definition (3.13). Degenerations at $q \to 1$ of the elements $\psi_\Lambda \in G'_n(q)$ were studied in [**13**]. In the subsequent two sections we will give q-analogues of these results from [**13**].

5 On the Divisibility of the Element ψ_{Λ^r}

The opening part of this section examines the left-divisibility of the element $\psi_{\Lambda^r} \in G'_n(q)$ by certain elements in the algebra $G'_n(q)$ corresponding to pairs of adjacent row entries in Λ^r. In the next proposition, the scalars q_k and q_{k+1} are defined by (4.6) for $\Lambda = \Lambda^r$.

Proposition 5.1. *Suppose that* $k = \Lambda^r(i,j)$ *and* $k+1 = \Lambda^r(i,j+1)$ *are adjacent entries in some row of* Λ^r. *Then* $\psi_k(q_k, q_{k+1}) \cdot \psi_{\Lambda^r} \;=\; 0$.

Proof. For any n-tuple $(x_1, \ldots, x_n)$ in $\mathcal{X} \cap \mathcal{F}$ we have $\psi_{\Lambda^r}(x_1, \ldots, x_n) \;=\; \pi_\chi(\Phi_{w_{\Lambda^r}})(1)$ where χ is the generic character determined by $x_1, \ldots, x_n$ through (3.12) with $\Lambda = \Lambda^r$. Now the entry k precedes $k+1$ in the column sequence $(\Lambda^r)^*$. By definition of the permutation $w_{\Lambda^r} \in S_n$, it follows that $\mathrm{length}(w_{\Lambda^r}) \;=\; \mathrm{length}(s_k w_{\Lambda^r}) + 1$. Then Proposition 3.3 gives

$$
\begin{aligned}
\pi_\chi(\Phi_{w_{\Lambda^r}})(1) \;&=\; \pi_{s_k w_{\Lambda^r} \cdot \chi}(\Phi_k)(1) \cdot \pi_\chi(\Phi_{s_k w_{\Lambda^r}})(1) \\
&=\; \psi_k(x_{k+1}, x_k) \cdot \pi_\chi(\Phi_{s_k w_{\Lambda^r}})(1)
\end{aligned}
\tag{5.1}
$$

where the second equality is given by (3.14). Furthermore, by the definition of the set $\mathcal{F}$, the pair (x_{k+1}, x_k) satisfies the idempotency condition (4.1); thus the first relation in Lemma 4.1 gives the equality $\psi_k(x_k, x_{k+1}) \cdot \psi_k(x_{k+1}, x_k) \;=\; 0$. It follows from (5.1) that

$$
\psi_k(x_k, x_{k+1}) \cdot \psi_{\Lambda^r}(x_1, \ldots, x_n) \;=\; 0\,, \qquad (x_1, \ldots, x_n) \in \mathcal{X} \cap \mathcal{F}. \tag{5.2}
$$

Since the restriction of the rational function on the left hand side of (5.2) to $\mathcal{F}$ is regular in $\mathcal{F} \cap \mathcal{Y}$ then the stated result follows by continuation along $\mathcal{F}$ to the point $(q_1, \ldots, q_n)$. ∎

An immediate consequence of (3.13) is

$$
\psi_k(y, x) \;=\; \psi_k(x, y) \;+\; \varepsilon \cdot \frac{x+y}{x-y}\,. \tag{5.3}
$$

Using this identity we obtain the following corollary to Proposition 5.1.

Corollary 5.2. *Suppose that* $k = \Lambda^r(i,j)$ *and* $k+1 = \Lambda^r(i,j+1)$ *are adjacent entries in some row of the tableau* Λ^r. *Then*

$$
\psi_k(q_{k+1}, q_k) \cdot \psi_{\Lambda^r} \;=\; \varepsilon \cdot \frac{q_k + q_{k+1}}{q_k - q_{k+1}} \cdot \psi_{\Lambda^r}\,.
$$

An analogue of Proposition 5.1 exists for adjacent column entries in the tableau Λ^r, but it is not as apparent: given any n-tuple $(x_1, \ldots, x_n)$ in $\mathcal{F}$, the pair (x_k, x_l) where k and l are adjacent entries in the same column of Λ^r does not necessarily satisfy the idempotency condition. The remainder of this section is devoted to proving this analogue (Corollary 5.8).

Once again, consider the restriction of the rational function $\psi_k(x,y)\,\psi_{k+1}(z,y)\,\psi_k(z,x)$ to the set $\mathcal{I}$. Lemma 4.3 demonstrates that this restriction is regular at $z = y$ and has identical values to the function $\theta_k(x,y)$. This expression is not necessarily divisible on the right by $\psi_k(y,x)$. However, let us consider the rational function

$$
\psi_k(x,y)\,\psi_{k+1}(z,y)\,\psi_k(z,x) \cdot \psi_k(x,z) \tag{5.4}
$$

and denote by $d(x,y)$ the rational function

$$
\varepsilon^3 \cdot y \left\{ (y^2 - 1) \left(\frac{x^3}{(xy-1)^4} + \frac{x^{-3}}{(x^{-1}y-1)^4} \right) + \frac{x^3 - x}{(xy-1)^4} + \frac{x^{-3} - x^{-1}}{(x^{-1}y-1)^4} \right\}
$$

valued in $\mathbb{K}$. We will need the following lemma.

Lemma 5.3. *The restriction of the function* (5.4) *to* $\mathcal{I}$ *coincides at* $z = y$ *with the restriction of the product* $d(x,y)\,\psi_k(x,y)$.

Proof. First, let us examine the product $\psi_k(z,x)\,\psi_k(x,z)$. The first relation in Lemma 4.1 and the idempotency condition on (x,y) give the equality

$$\psi_k(z,x)\,\psi_k(x,z) \;=\; \varepsilon^2\cdot\left(\frac{xy}{(xy-1)^2}+\frac{x^{-1}y}{(x^{-1}y-1)^2}-\frac{xz}{(xz-1)^2}-\frac{x^{-1}z}{(x^{-1}z-1)^2}\right)$$

on $\mathcal{I}$. A direct calculation shows that the restriction of (5.4) to $\mathcal{I}$ can be written as

$$\psi_k(x,y)\left(T_{k+1}+\frac{\varepsilon}{z^{-1}y-1}+\frac{\varepsilon}{zy-1}\,C_{k+1}C_{k+2}\right)\cdot\varepsilon^2\cdot\Big\{(yz-1)(y^{-1}z-1)\Big\}\,y\cdot\,d(x,y,z)$$

where $d(x,y,z)$ denotes the sum

$$\frac{x^3}{(xy-1)^2(xz-1)^2}+\frac{x^3-x}{(xy-1)^2(xz-1)^2(yz-1)}+$$
$$\frac{x^{-3}}{(x^{-1}y-1)^2(x^{-1}z-1)^2}+\frac{x^{-3}-x^{-1}}{(x^{-1}y-1)^2(x^{-1}z-1)^2(yz-1)}\,.$$

Thus the restriction to $\mathcal{I}$ of (5.4) is regular at $z=y$ provided that $y^2\neq 1$; moreover, it takes identical values to the function

$$\varepsilon^3\cdot d(x,y,y)\;y(y^2-1)\cdot\psi_k(x,y)\;=$$
$$\varepsilon^3\cdot y\left\{(y^2-1)\left(\frac{x^3}{(xy-1)^4}+\frac{x^{-3}}{(x^{-1}y-1)^4}\right)+\frac{x^3-x}{(xy-1)^4}+\frac{x^{-3}-x^{-1}}{(x^{-1}y-1)^4}\right\}\cdot\psi_k(x,y)\,.$$

As a consequence, the restriction is also regular at $y^2=1$. ■

Corollary 5.4. *The restriction of* (5.4) *to* $\mathcal{I}$ *vanishes at* $y=z=1$.

Proof. On the set $\mathcal{I}$, the function $d(x,y)$ takes the value zero when $y=1$. ■

We will use Corollary 5.4 with $y=x_k$ and $z=x_l$ where both k and l stand on the *leading diagonal* of the tableau Λ: this is the diagonal with the entries $\Lambda(i,i)$. At the special point $(q_1,\dots,q_n)$ the definition (4.6) then gives $q_k=q_l=1$. Next, let us consider the involutive antiautomorphism $\alpha: G_n'(q)\to G_n'(q)$ defined by $T_k\mapsto T_{n-k}$ and $C_k\mapsto C_{n-k+1}$. Note that $\alpha\big(\psi_k(x,y)\big) \;=\; \psi_{n-k}(x,y)$ for each index $k=1,\dots,n-1$. The next result concerns the element ψ_{Λ^c} associated with the column tableau.

Proposition 5.5. *The element* $\psi_{\Lambda^c}\in G_n'(q)$ *is invariant under the antiautomorphism* α.

Proof. For the column tableau Λ^c, each sequence $\mathcal{B}_k^*$ is the complete interval $(1,2,\dots,k-1)$. Hence the equality of rational functions (4.7) becomes

$$\psi_{\Lambda^c}(x_1,\dots,x_n)\;=\;\prod_{k=2,\dots,n}^{\longrightarrow}\left(\prod_{j=1,\dots,k-1}^{\longrightarrow}\psi_{k-j}(x_k,x_j)\right).$$

A direct application of the second and third equalities in Lemma 4.1 establishes that

$$\psi_{\Lambda^c}(x_1,\dots,x_n)\;=\;\prod_{k=2,\dots,n}^{\longleftarrow}\left(\prod_{j=1,\dots,k-1}^{\longleftarrow}\psi_{n-k+j}(x_k,x_j)\right)\;=\;\alpha\left(\psi_{\Lambda^c}(x_1,\dots,x_n)\right)\,.$$

In particular, the restriction of the function $\psi_{\Lambda^c}(x_1,\ldots,x_n)$ onto $\mathcal{X}\cap\mathcal{F}$ is invariant under α. Now Proposition 5.5 follows by continuation along $\mathcal{F}$ to the point $(q_1,\ldots,q_n)$. ■

In the proof of Theorem 4.4 we showed that the restriction of the function $\theta_{\Lambda^c}(x_1,\ldots,x_n)$ to $\mathcal{F}$ is regular at the point $(q_1,\ldots,q_n)$. Define the element $\theta_{\Lambda^c}\in G_n'(q)$ as the value of the restriction at this point. We now present the main result in this section; here q_k and q_{k+1} are defined by (4.6) for $\Lambda=\Lambda^c$.

Theorem 5.6. *Suppose that $k=\Lambda^c(i,j)$ and $k+1=\Lambda^c(i+1,j)$ are adjacent entries in some column of the column tableau. Then $\theta_{\Lambda^c}\in G_n'(q)$ is divisible on the left by $\psi_k(q_{k+1},q_k)$.*

Proof. First, observe that Theorem 5.6 is implied by its particular case where $k=n-1$. Indeed, given an arbitrary index k, let Ω^c be the tableau obtained from Λ^c by removing each of the symbols $k+2,\ldots,n$. Then the tableau Ω^c is the column tableau for a certain partition $\omega\succ k+1$ and $\theta_{\Lambda^c}=\theta_{\Omega^c}\cdot\tilde\theta$ for some element $\tilde\theta\in G_n'(q)$.

Thus we will assume that $k=n-1$. Let us demonstrate left divisibility of the element θ_{Λ^c} by $\psi_{n-1}(q_n,q_{n-1})$. In particular, we will establish the equality

$$\psi_{n-1}(q_n,q_{n-1})\cdot\theta_{\Lambda^c} \;=\; \varepsilon\cdot\frac{q_{n-1}+q_n}{q_{n-1}-q_n}\cdot\theta_{\Lambda^c}\,. \tag{5.5}$$

By (5.3), this is equivalent to verifying

$$\psi_{n-1}(q_{n-1},q_n)\cdot\theta_{\Lambda^c} \;=\; 0\,. \tag{5.6}$$

Recall that $n-1=\Lambda^c(i,j)$ and $n=\Lambda^c(i+1,j)$. Let $p_1<p_2<\cdots<p_{j-i}=n$ be the entries in the $(i{+}1)$-th row of the tableau Λ^c. The restriction to $\mathcal{F}$ of the factor $\theta'_{\Lambda^c}(x_1,\ldots,x_n)$ in (4.9) is regular at the point $(q_1,\ldots,q_n)$; let us denote its value by $\theta'_{\Lambda^c}\in G_n'(q)$. Using the second and third equations in Lemma 4.1, this element can be shown to satisfy

$$\theta'_{\Lambda^c}\cdot\psi_1(q_{n-1},q_n) \;=\; \prod_{k=1,\ldots,j-i}^{\longrightarrow}\psi_{j-i+n-b_n-k}(q_{n-1},q_{p_k})\;\cdot\;\eta$$

for some element $\eta\in G_n'(q)$; cf. [**14**, Proposition 2.8]. Then

$$\psi_{\Lambda^c}\cdot\psi_1(q_{n-1},q_n) \;=\; \theta_{\Lambda^c}\cdot\prod_{k=1,\ldots,j-i}^{\longrightarrow}\psi_{j-i+n-b_n-k}(q_{n-1},q_{p_k})\;\cdot\;\eta$$

and it will be sufficient to verify that

$$\theta_{\Lambda^c}\cdot\prod_{k=1,\ldots,j-i}^{\longrightarrow}\psi_{j-i+n-b_n-k}(q_{n-1},q_{p_k}) \;=\; 0\,. \tag{5.7}$$

Applying the antiautomorphism α to the equality $\psi_{\Lambda^c}\cdot\psi_1(q_{n-1},q_n)=0$ using Proposition 5.5, we obtain $\psi_{n-1}(q_{n-1},q_n)\cdot\psi_{\Lambda^c}=0$. Since the element ψ_{Λ^c} can be realised by multiplying θ_{Λ^c} on the right by the invertible element θ'_{Λ^c}, this last equality is equivalent to the required statement (5.6). It therefore remains to establish (5.7). This will be proved using induction on the integer $j-i$.

I. Suppose that $j-i=1$; that is, the symbol n stands on the leading diagonal in the column tableau Λ^c. Then $b_n=n-1$. Let $m=\Lambda^c(i,i)$ then $\mathcal{B}_n(n-2)=m$ and $\mathcal{B}_n(n-1)=n-1$. The expression (4.10) defining the function $\theta_{\Lambda^c}(x_1,\ldots,x_n)$ takes the form of the product

$$\theta(x_1,\ldots,x_n)\cdot\psi_2(x_n,x_m)\,\psi_1(x_n,x_{n-1})$$

where

$$\theta(x_1,\dots,x_n)=\prod^{\rightarrow}_{k=2,\dots,n-1}\left(\prod^{\rightarrow}_{p=1,\dots,b_k}\psi_{k-p}(x_k,x_{\mathcal{B}_k(p)})\right)\times\prod^{\rightarrow}_{p=1,\dots,n-3}\psi_{n-p}(x_n,x_{\mathcal{B}_n(p)})$$

on restriction to $\mathcal{F}$ is regular at $(q_1,\dots,q_n)$. Due to Lemma 4.1 and Corollary 4.2(a), this restriction satisfies the equality

$$\theta(x_1,\dots,x_n)\;=\;\varepsilon^{-1}\cdot\frac{x_m-x_{n-1}}{x_m+x_{n-1}}\cdot\theta(x_1,\dots,x_n)\cdot\psi_1(x_{n-1},x_m)\,.$$

Thus the left hand side in (5.7) is the value at $(q_1,\dots,q_n)$ taken by the restriction of

$$\varepsilon^{-1}\cdot\frac{x_m-x_{n-1}}{x_m+x_{n-1}}\cdot\theta(x_1,\dots,x_n)\times\psi_1(x_{n-1},x_m)\,\psi_2(x_n,x_m)\,\psi_1(x_n,x_{n-1})\cdot\psi_1(x_{n-1},x_n)\,.$$

For $(x_1,\dots,x_n)\in\mathcal{F}$, the pair (x_{n-1},x_m) satisfies the condition (4.1) with $x_m+x_{n-1}\neq 0$; while $q_m=q_n=1$. Corollary 5.4 demonstrates that the restriction vanishes at $(q_1,\dots,q_n)$.

II. Next, consider the case $j-i>1$. We will demonstrate that the restriction to $\mathcal{F}$ of

$$\theta_{\Lambda^c}(x_1,\dots,x_n)\;\cdot\prod^{\rightarrow}_{k=1,\dots,j-i}\psi_{j-i+n-b_n-k}(x_{n-1},x_{p_k})\tag{5.8}$$

has the value zero at $(q_1,\dots,q_n)$. In this instance, let us specify $m=\Lambda^c(i+1,j-1)$ then $m-1=\Lambda^c(i,j-1)$. Let Ω^c denote the tableau obtained from Λ^c by removing each of the entries $m+1,\dots,n$; evidently, Ω^c is the column tableau for a certain partition $\omega\succ m$. Now the expression in (4.10) defining the function $\theta_{\Lambda^c}(x_1,\dots,x_n)$ can be expanded as

$$\begin{aligned}&\theta_{\Omega^c}(x_1,\dots,x_n)\cdot\theta'(x_1,\dots,x_n)\,\psi_{n-b_{n-1}-1}(x_{n-1},x_{m-1})\;\;\times\\&\quad\theta''(x_1,\dots,x_n)\,\psi_{n-b_n+j-i}(x_n,x_{m-1})\,\psi_{n-b_n+j-i-1}(x_n,x_{n-1})\;\theta(x_1,\dots,x_n)\end{aligned}$$

where the functions $\theta'(x_1,\dots,x_n)$ and $\theta''(x_1,\dots,x_n)$ are defined by

$$\begin{aligned}\theta'(x_1,\dots,x_n)\;&=\prod^{\rightarrow}_{k=m+1,\dots,n-2}\left(\prod^{\rightarrow}_{p=1,\dots,b_k}\psi_{k-p}(x_k,x_{\mathcal{B}_k(p)})\right)\times\prod^{\rightarrow}_{p=1,\dots,b_{n-1}-1}\psi_{n-1-p}(x_{n-1},x_{\mathcal{B}_{n-1}(p)}),\\ \theta''(x_1,\dots,x_n)\;&=\prod^{\rightarrow}_{p=1,\dots,b_n-j+i-1}\psi_{n-p}(x_n,x_{\mathcal{B}_n(p)})\end{aligned}$$

while

$$\theta(x_1,\dots,x_n)\;=\prod^{\rightarrow}_{k=1,\dots,j-i-1}\psi_{n-b_n+j-i-1-k}(x_n,x_{p_k})\,.$$

Since the integers b_{n-1} and b_n are related by the equality $b_n=b_{n-1}+j-i$, we have the inequalities $n-p\;\geq\;n-b_{n-1}+1$ for each $p=1,\dots,b_n-j+i-1$. Thus the function $\psi_{n-b_{n-1}-1}(x_{n-1},x_{m-1})$ commutes with $\theta''(x_1,\dots,x_n)$ by the second relation in Lemma 4.1. Therefore the function (5.8) can be written as

$$\begin{aligned}&\theta_{\Omega^c}(x_1,\dots,x_n)\cdot\theta'(x_1,\dots,x_n)\,\theta''(x_1,\dots,x_n)\;\;\times\\&\quad\psi_{n-b_{n-1}-1}(x_{n-1},x_{m-1})\,\psi_{n-b_{n-1}}(x_n,x_{m-1})\,\psi_{n-b_{n-1}-1}(x_n,x_{n-1})\\&\quad\times\;\theta(x_1,\dots,x_n)\;\cdot\prod^{\rightarrow}_{k=1,\dots,j-i}\psi_{n-b_{n-1}-k}(x_{n-1},x_{p_k})\end{aligned}$$

while further use of the second and third equalities in Lemma 4.1 gives the expression

$$
\begin{aligned}
&\theta_{\Omega^c}(x_1,\dots,x_n)\cdot\theta'(x_1,\dots,x_n)\,\theta''(x_1,\dots,x_n) \quad\times\\
&\psi_{n-b_{n-1}-1}(x_{n-1},x_{m-1})\,\psi_{n-b_{n-1}}(x_n,x_{m-1})\,\psi_{n-b_{n-1}-1}(x_n,x_{n-1})\,\psi_{n-b_{n-1}-1}(x_{n-1},x_n)\\
&\times \prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-1-k}(x_{n-1},x_{p_k})\prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-k}(x_n,x_{p_k})\,.
\end{aligned}
\tag{5.9}
$$

It follows from the proof of Theorem 4.4 that the product in the first line of (5.9) on restriction to $\mathcal{F}$ is regular at $(q_1,\dots,q_n)$. The restriction of each of the factors in the last line is also regular at this point. Furthermore, for any $(x_1,\dots,x_n)\in\mathcal{F}$, the pair (x_{n-1},x_{m-1}) satisfies the condition (4.1). Thus Lemma 5.3 demonstrates that the restriction of the function on the second line in (5.9) coincides at $(q_1,\dots,q_n)$ with the restriction to $\mathcal{F}$ of the function

$$
d(x_{n-1},x_{m-1})\cdot\psi_{n-b_{n-1}-1}(x_{n-1},x_{m-1}).
$$

Hence the restriction of (5.9) to $\mathcal{F}$ has the same value at $(q_1,\dots,q_n)$ as the restriction of

$$
\begin{aligned}
&\theta_{\Omega^c}(x_1,\dots,x_n)\cdot\theta'(x_1,\dots,x_n)\,\theta''(x_1,\dots,x_n)\cdot d(x_{n-1},x_n)\,\psi_{n-b_{n-1}-1}(x_{n-1},x_n) \quad\times\\
&\qquad\prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-1-k}(x_{n-1},x_{p_k})\prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-k}(x_n,x_{p_k})\\
=\;&\theta_{\Omega^c}(x_1,\dots,x_n)\cdot\theta'(x_1,\dots,x_n)\,\theta''(x_1,\dots,x_n)\cdot\prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-1-k}(x_n,x_{p_k}) \quad\times\\
&\qquad\prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-k}(x_{n-1},x_{p_k})\cdot d(x_{n-1},x_n)\,\psi_{n-b_n}(x_{n-1},x_n)\,.
\end{aligned}
$$

Replacing the variable x_n by x_{m-1} in every factor $\psi_{n-b_{n-1}-1-k}(x_n,x_{p_k})$ within the fourth component of the latter expression does not affect the value of the restriction at $(q_1,\dots,q_n)$. Let us denote this modified component by

$$
\bar\theta(x_1,\dots,x_n) \;=\; \prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{n-b_{n-1}-1-k}(x_{m-1},x_{p_k})\,.
$$

For each index $k=1,2,\dots,j-i-1$, we have the inequalities $(n-b_{n-1}-1-k)+2\;<\;n-p$ for every $1\le p\le b_n-j+i-1$; thus $\bar\theta(x_1,\dots,x_n)$ commutes with $\theta''(x_1,\dots,x_n)$.

The proof is complete once we demonstrate that the restriction to $\mathcal{F}$ of

$$
\theta_{\Omega^c}(x_1,\dots,x_n)\,\theta'(x_1,\dots,x_n)\,\bar\theta(x_1,\dots,x_n)
\tag{5.10}
$$

vanishes at the point $(q_1,\dots,q_n)$. The factors in the function $\theta'(x_1,\dots,x_n)$ are arranged with respect to the ordering specified by the sequences $\mathcal{B}_k$. Let us now rearrange these factors in the following manner: for each index $k>m$ appearing in the $(j-1)$-th column of Λ^c, change the subsequence $m-1\,,\,p_1\,,\,p_2\,,\dots,\,p_{j-i-1}$ in $\mathcal{B}_k$ to $p_1\,,\,p_2\,,\dots,\,p_{j-i-1}\,,\,m-1$. Recall that

$$
m-1=\Lambda^c(i,j-1)\,,\quad p_1=\Lambda^c(i+1,i+1)\,,\quad \dots,\quad p_{j-i-1}=\Lambda^c(i+1,j-1).
$$

Let $\theta^*(x_1,\dots,x_n)$ denote the product obtained from $\theta'(x_1,\dots,x_n)$ by this rearrangement; put

$$
\tilde\theta(x_1,\dots,x_n) \;=\; \prod_{k=1,\dots,j-i-1}^{\rightarrow}\psi_{m-b_{n-1}+i-k}(x_{m-1},x_{p_k})\,.
$$

The equality

$$\theta'(x_1,\dots,x_n)\cdot\bar\theta(x_1,\dots,x_n) \;=\; \tilde\theta(x_1,\dots,x_n)\cdot\theta^*(x_1,\dots,x_n)$$

of rational functions can be established by using the second and third relations in Lemma 4.1. Hence the product (5.10) becomes $\theta_{\Omega^c}(x_1,\dots,x_n)\,\tilde\theta(x_1,\dots,x_n)\,\theta^*(x_1,\dots,x_n)$. Furthermore, any singular factors in the restriction of the function $\theta^*(x_1,\dots,x_n)$ to $\mathcal{F}$ can be dealt with as described in the proof of Theorem 4.4; that is, we consider an expression $\hat\theta^*(x_1,\dots,x_n)$ obtained by inserting normalised factors into $\theta^*(x_1,\dots,x_n)$ at certain positions. Then

$$\theta_{\Omega^c}(x_1,\dots,x_n)\,\tilde\theta(x_1,\dots,x_n)\,\theta^*(x_1,\dots,x_n) \;=\; \theta_{\Omega^c}(x_1,\dots,x_n)\,\tilde\theta(x_1,\dots,x_n)\,\hat\theta^*(x_1,\dots,x_n)$$

on $\mathcal{F}$, while the restriction of $\hat\theta^*(x_1,\dots,x_n)$ to $\mathcal{F}$ is regular at $(q_1,\dots,q_n)$. Meanwhile, the inductive assumption on the tableau Ω^c shows that the restriction to $\mathcal{F}$ of the function

$$\theta_{\Omega^c}(x_1,\dots,x_n)\cdot\tilde\theta(x_1,\dots,x_n) \;=\; \theta_{\Omega^c}(x_1,\dots,x_n)\;\cdot \prod_{k=1,\dots,j-i-1}^{\longrightarrow} \psi_{m-b_m+j-1-i-k}(x_{m-1},x_{p_k})$$

vanishes at this special point; thus the restriction of (5.10) also vanishes. ■

The next result is a consequence of Proposition 5.5 and Theorem 5.6. Here q_k and q_{k+1} are again defined by (4.6) for the tableau $\Lambda=\Lambda^c$.

Corollary 5.7. *Suppose that $k=\Lambda^c(i,j)$ and $k+1=\Lambda^c(i+1,j)$ are adjacent entries within the same column of Λ^c. Then the element $\psi_{\Lambda^c}\in G'_n(q)$ is:*
a) divisible on the left by $\psi_k(q_{k+1},q_k)$ and
b) divisible on the right by $\psi_{n-k}(q_{k+1},q_k)$.

Furthermore, we obtain the following analogue to Corollary 5.2. In this instance, we specify q_k and q_l by (4.6) for the row tableau $\Lambda=\Lambda^r$.

Corollary 5.8. *Suppose that $k=\Lambda^r(i,j)$ and $l=\Lambda^r(i+1,j)$ are adjacent entries within the same column of the row tableau. Then*

$$\psi_{\Lambda^r}\cdot\psi_{n-p}(q_l,q_k) \;=\; \varepsilon\cdot\frac{q_k+q_l}{q_k-q_l}\cdot\psi_{\Lambda^r}$$

where $p=\Lambda^c(i,j)$ is the entry in the column tableau occupying the same position as k in Λ^r.

Proof. Since $p=\Lambda^c(i,j)$ then $p+1=\Lambda^c(i+1,j)$. Thus Corollary 5.7(b) yields the equality

$$\psi_{\Lambda^c}\cdot\psi_{n-p}(q'_{p+1},q'_p) \;=\; \varepsilon\cdot\frac{q'_p+q'_{p+1}}{q'_p-q'_{p+1}}\cdot\psi_{\Lambda^c}$$

where q'_p and q'_{p+1} are defined by (4.6) for $\Lambda=\Lambda^c$. The identification $q'_p=q_k\,,\ q'_{p+1}=q_l$ is obvious. Meanwhile, since $\psi_{\Lambda^r}=\mu\cdot\psi_{\Lambda^c}$ for some invertible element $\mu\in G'_n(q)$ then ψ_{Λ^c} may be replaced by ψ_{Λ^r} in the above equality to give the stated result. ■

The element $\psi_{\Lambda^r}T^{-1}_{w_{\Lambda^r}}\in G'_n(q)$ is our analogue of the product $E_\omega\in H_n(q)$ as described in Section 1. We will justify this claim further in the next section. We also conjecture that this element of $G'_n(q)$ is an idempotent up to a multiplier from $\mathbb{F}^*$. Using Propositions 6.3 and 6.5 one can prove that the degeneration at $q\to1$ of this element of $G'_n(q)$ is an idempotent in the algebra G_n up to a certain multiple from $\mathbb{C}^*$. This confirms [**13**, Conjecture 9.4].

6 The $G_n'(q)$-Module V_λ

In this final section for each partition $\lambda \succ n$ we will construct a certain $G_n'(q)$-module V_λ. The $G_n^-(q)$-module $V_\lambda \otimes_{\mathbb{F}} \mathbb{K}$ will appear as a subrepresentation in the principal series representation M_{χ_0} of $Se_n(q)$. Here χ_0 is the character of $\mathbb{C}(q)[X]$ determined through any shifted tableau $\Lambda \in \mathcal{T}_\lambda$ by $w_\Lambda \cdot \chi_0(X_k^{-1}) = q_k$, see the definition (4.6). The subalgebra $G_n(q) \subset Se_n(q)$ acts in $M_{\chi_0} = G_n^-(q)$ via left multiplication. We will observe that the action of $Se_n(q)$ in $V_\lambda \otimes_{\mathbb{F}} \mathbb{K}$ factors through the homomorphism $\iota : Se_n(q) \to G_n(q)$ introduced in Proposition 3.5. The $G_n'(q)$-module V_λ is reducible; its irreducible components are described in Theorem 6.7.

Consider the partial *Bruhat order* $\succ$ on the elements in the symmetric group S_n: given permutations $s, s' \in S_n$ then $s \succ s'$ if and only if there are adjacent transpositions $s_{k_1}, \ldots, s_{k_p}$ such that $s = s_{k_p} \cdots s_{k_1} \cdot s'$ where $\mathrm{length}(s) = \mathrm{length}(s') + p$. For any $\Lambda \in \mathcal{S}_\lambda$ the elements w_Λ and $s_k w_\Lambda$ in S_n are neighbours with respect to this partial ordering for each $k = 1, \ldots, n-1$. We now examine further the elements $\psi_\Lambda \in G_n'(q)$ introduced in Section 4.

Proposition 6.1. *Let $\Lambda \in \mathcal{S}_\lambda$ and $k \in \{1, 2, \ldots, n-1\}$. Define $q_k, q_{k+1} \in \mathbb{F}$ by (4.6). Then*
a) if $s_k \cdot \Lambda \in \mathcal{S}_\lambda$ and $s_k w_\Lambda \succ w_\Lambda$ then $\psi_k(q_k, q_{k+1}) \cdot \psi_\Lambda = \psi_{s_k \cdot \Lambda}$;
b) if $s_k \cdot \Lambda \in \mathcal{S}_\lambda$ and $s_k w_\Lambda \prec w_\Lambda$ then $\psi_k(q_{k+1}, q_k) \cdot \psi_{s_k \cdot \Lambda} = \psi_\Lambda$;
c) if $s_k \cdot \Lambda \notin \mathcal{S}_\lambda$ then $\psi_k(q_k, q_{k+1}) \cdot \psi_\Lambda = 0$.

Proof. Let us fix the standard tableau $\Lambda \in \mathcal{S}_\lambda$ and the index k. The tableau $s_k \cdot \Lambda$ obtained by interchanging the entries k and $k+1$ within Λ, may or may not be standard. Suppose that $k = \Lambda(i, j)$ and $k+1 = \Lambda(i', j')$. Since the tableau Λ is standard, we have four possibilities to consider:

$$i' > i,\ j' < j\,; \qquad i' < i,\ j' > j\,; \qquad i' = i,\ j' = j+1\,; \qquad i' = i+1,\ j' = j.$$

The tableau $s_k \cdot \Lambda$ is standard in precisely the first two cases. Furthermore, using the definition of the permutation $w_\Lambda \in S_n$ gives $w_{s_k \cdot \Lambda} \succ w_\Lambda$ in the first situation while $w_{s_k \cdot \Lambda} \prec w_\Lambda$ in the second instance. We will now examine each case separately.

i) In the first case, we have $s_k \cdot \Lambda \in \mathcal{S}_\lambda$ and $s_k w_\Lambda \succ w_\Lambda$. Let us verify the equality in (a) for the elements ψ_Λ and $\psi_{s_k \cdot \Lambda}$. Given any n-tuple $(x_1, \ldots, x_n) \in \mathcal{X} \cap \mathcal{F}$, we have

$$\psi_\Lambda(x_1, \ldots, x_n) \;=\; \pi_\chi(\Phi_{w_\Lambda})(1)$$

where χ is the generic character determined through (3.12) by $x_1, \ldots, x_n$. Similarly for the tableau $s_k \cdot \Lambda$, we have the equality

$$\psi_{s_k \cdot \Lambda}(x_1, \ldots, x_n) \;=\; \pi_\chi(\Phi_{s_k w_\Lambda})(1)$$

on $\mathcal{X} \cap \mathcal{F}$. Using Proposition 3.3, the restriction of $\psi_{s_k \cdot \Lambda}(x_1, \ldots, x_n)$ to $\mathcal{F}$ can be written as

$$\pi_{w_\Lambda \cdot \chi}(\Phi_k)(1) \cdot \pi_\chi(\Phi_{w_\Lambda})(1) \;=\; \psi_k(x_k, x_{k+1}) \cdot \psi_\Lambda(x_1, \ldots, x_n).$$

Here the equality follows from (3.14) since $w_\Lambda \cdot \chi(X_k^{-1}) = x_k\,,\ w_\Lambda \cdot \chi(X_{k+1}^{-1}) = x_{k+1}$. The stated result is obtained by continuation along $\mathcal{F}$ to the point $(q_1, \ldots, q_n)$.

ii) The conditions in (b) are realised in the second case: $s_k \cdot \Lambda \in \mathcal{S}_\lambda$ and $s_k w_\Lambda \prec w_\Lambda$. Let us consider the tableau $\Lambda' = s_k \cdot \Lambda$. Then $s_k \cdot \Lambda' = \Lambda$ while $s_k w_{\Lambda'} \succ w_{\Lambda'}$. The result follows from (a) for the tableau Λ'.

iii) In the third instance, the entries k and $k+1$ are adjacent in some row of Λ and hence $s_k \cdot \Lambda \notin \mathcal{S}_\lambda$. The equality in (c) has been verified for the row tableau Λ^r in Proposition 5.1; this proof extends directly to an arbitrary tableau $\Lambda \in \mathcal{S}_\lambda$.

iv) In the final case, the equality in (c) has been established for the column tableau Λ^c in the preceding section. Let us now extend this equality to an arbitrary tableau $\Lambda \in \mathcal{S}_\lambda$. The entries $k = \Lambda(i,j)$ and $k+1 = \Lambda(i+1,j)$ are adjacent within some column of Λ. Since the symbol k precedes $k+1$ in the column sequence $(\Lambda)^*$ then $s_k w_\Lambda \prec w_\Lambda$ by the definition of the element $w_\Lambda \in S_n$; it also follows from Lemma 2.3 that $s_\Lambda s_k \succ s_\Lambda$. Furthermore, the definition of the permutation s_Λ gives $s_\Lambda s_k = s_p s_\Lambda$ where $p = \Lambda^c(i,j)$ and $p+1 = \Lambda^c(i+1,j)$ are the entries in Λ^c occupying the same positions as k and $k+1$ respectively, in the tableau Λ. In particular, we have $\mathrm{length}(s_p s_\Lambda) = \mathrm{length}(s_\Lambda s_k) = \mathrm{length}(s_\Lambda) + 1$.

For any generic character χ, the element $\pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda s_k})(1) \in G_n'(q)$ can be expressed as

$$\pi_{s_k w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \cdot \pi_{w_\Lambda \cdot \chi}(\Phi_k)(1) \;=\; \pi_{s_k w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \cdot \psi_k(x_k, x_{k+1})$$

using Proposition 3.3 and the equality (3.14). Note that x_k and x_{k+1} are determined by χ via (3.12) for the tableau Λ. The same element $\pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda s_k})(1)$ may also be expressed as

$$\begin{aligned}\pi_{s_\Lambda w_\Lambda \cdot \chi}(\Phi_p)(1) \cdot \pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \;&=\; \pi_{w_0 \cdot \chi}(\Phi_p)(1) \cdot \pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1)\\ &=\; \psi_p(x_k, x_{k+1}) \cdot \pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1)\end{aligned}$$

where the second equality is given by (3.14) since $w_0 \cdot \chi(X_p^{-1}) = x_k$ and $w_0 \cdot \chi(X_{p+1}^{-1}) = x_{k+1}$. Thus we have verified the identity

$$\psi_p(x_k, x_{k+1}) \cdot \pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \;=\; \pi_{s_k w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \cdot \psi_k(x_k, x_{k+1})$$

for any generic χ. Meanwhile the equality (3.15) gives

$$\psi_{\Lambda^c}(x_1, \ldots, x_n) \;=\; \pi_{w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \cdot \psi_\Lambda(x_1, \ldots, x_n)$$

for any n-tuple $(x_1, \ldots, x_n) \in \mathcal{X}$. Combining these two equalities on $\mathcal{X}$, we obtain

$$(6.1) \qquad \psi_p(x_k, x_{k+1}) \cdot \psi_{\Lambda^c}(x_1, \ldots, x_n) \;=\; \pi_{s_k w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1) \cdot \psi_k(x_k, x_{k+1}) \cdot \psi_\Lambda(x_1, \ldots, x_n)\,.$$

In the proof of Theorem 5.6, we have established that the restriction to $\mathcal{F}$ of the rational function on the left hand side in (6.1) vanishes at the point $(q_1, \ldots, q_n)$. Furthermore, it can be shown that the factor $\pi_{s_k w_\Lambda \cdot \chi}(\Phi_{s_\Lambda})(1)$ on the right hand side in (6.1) is regular in $\mathcal{Y}$ and has invertible values; cf. [**13**, Proposition 7.1]. Thus the equality in (c) follows by the continuation of (6.1) along $\mathcal{F}$ to the special point $(q_1, \ldots, q_n)$. ■

Now let $V_\lambda = G_n'(q)\, \psi_{\Lambda^r}$ be the left ideal in the algebra $G_n'(q)$ generated by the element ψ_{Λ^r}. Consider $G_n^{\sim}(q)$ as the space of the principal series representation M_{χ_0} of the affine Sergeev algebra $Se_n(q)$. Theorem 6.2 will demonstrate that this action of the affine algebra in $G_n^{\sim}(q)$ preserves the subspace $V_\lambda \otimes_{\mathbb{F}} \mathbb{K}$. Before stating this result, let us introduce some additional notation. The Clifford algebra, now viewed as the subalgebra in $G_n'(q)$ generated over $\mathbb{F}$ by the elements $C_1, \ldots, C_n$ has the natural basis $\mathcal{C}$ described in (2.3). For each element $C = C_{k_1} \cdots C_{k_p} \in \mathcal{C}$ let us define the function $\nu_C : \{1, 2, \ldots, n\} \to \{\, 1, -1 \,\}$ by

$$\nu_C(k) \;=\; \begin{cases} +1 & \text{if the word } C \text{ contains the letter } C_k\,; \\ -1 & \text{otherwise.} \end{cases}$$

Proposition 3.5 describes a homomorphism $\iota : Se_n(q) \to G_n(q)$ which is identical on the subalgebra $G_n(q)$. The part (c) of the following theorem shows that the action of the affine algebra $Se_n(q)$ in the subspace $V_\lambda \otimes_{\mathbb{F}} \mathbb{K} \subset M_{\chi_0}$ factors through this homomorphism.

Theorem 6.2. *a) The elements $C\psi_\Lambda$ where $C \in \mathcal{C}$ and $\Lambda \in \mathcal{S}_\lambda$ form a $\mathbb{F}$-basis in V_λ.*
b) For each $k = 1, 2, \ldots, n$, the action of the element X_k on the basis vectors is given by

$$\pi_{\chi_0}(X_k)(C\psi_\Lambda) \;=\; q_k^{\nu_C(k)} \cdot C\psi_\Lambda \qquad \textit{for all } C \in \mathcal{C}, \Lambda \in \mathcal{S}_\lambda$$

where q_k is defined by (4.6) for Λ.
c) Actions of the elements X_k and $\iota(X_k)$ of $Se_n(q)$ in the subspace $V_\lambda \otimes_{\mathbb{F}} \mathbb{K} \subset M_{\chi_0}$ coincide.

Proof. Denote by V the subspace in $G_n'(q)$ spanned by the elements $C\psi_\Lambda$ where $C \in \mathcal{C}, \Lambda \in \mathcal{S}_\lambda$. In the proof of Theorem 4.4 the element ψ_Λ is expanded into a sum with leading term T_{w_Λ}. Thus the elements $C\psi_\Lambda$ where $C \in \mathcal{C}$ and $\Lambda \in \mathcal{S}_\lambda$ are linearly independent over $\mathbb{F}$ and form a basis in the space V. We will prove that $V = V_\lambda$.

Due to (4.8) every element ψ_Λ can be expressed as $\mu_\Lambda\, \psi_{\Lambda^r}$ for some invertible $\mu_\Lambda \in G_n'(q)$. So we have the inclusion $V \subseteq V_\lambda$. The opposite inclusion $V \supseteq V_\lambda$ will be established once we have proved that the action of the algebra $G_n(q)$ in $G_n'(q)$ via left multiplication preserves the subspace V. The action of the elements $C_1, \ldots, C_n$ preserves the subspace in V

$$\bigoplus_{C \in \mathcal{C}} \mathbb{F} \cdot C\psi_\Lambda \tag{6.2}$$

for every $\Lambda \in \mathcal{S}_\lambda$, and thus preserves the space V itself. We will use Proposition 6.1 to verify that the action of the elements $T_1, \ldots, T_{n-1}$ preserves V. Let a standard tableau $\Lambda \in \mathcal{S}_\lambda$ and an index $k \in \{1, \ldots, n-1\}$ be fixed. There are three cases to consider.

i) Suppose that $s_k \cdot \Lambda$ is standard and that $s_k w_\Lambda \succ w_\Lambda$. Then Proposition 6.1(a) gives the equality $\psi_k(q_k, q_{k+1}) \cdot \psi_\Lambda \;=\; \psi_{s_k \cdot \Lambda}$. Rewriting this equality in the form

$$T_k \cdot \psi_\Lambda \;=\; \psi_{s_k \cdot \Lambda} \,-\, \left(\frac{\varepsilon}{q_k^{-1} q_{k+1} - 1} + \frac{\varepsilon}{q_k q_{k+1} - 1}\, C_k C_{k+1} \right) \psi_\Lambda \tag{6.3}$$

explicitly describes the action of the generator T_k on the basis element ψ_Λ.

ii) Next, suppose that $s_k \cdot \Lambda$ is standard but that $s_k w_\Lambda \prec w_\Lambda$. Then Proposition 6.1(b) yields $\psi_k(q_{k+1}, q_k) \cdot \psi_{s_k \cdot \Lambda} \;=\; \psi_\Lambda$. Since $(q_1, \ldots, q_n) \in \mathcal{Y}$ then the pair (q_{k+1}, q_k) does not satisfy the condition (4.1); consequently the element $\psi_k(q_{k+1}, q_k)$ is invertible. Let us multiply the above equality on the left by the element $\psi_k(q_k, q_{k+1})$. Put

$$\beta_k = 1 \,-\, \varepsilon^2 \cdot \left(\frac{q_{k+1}^{-1} q_k}{(q_{k+1}^{-1} q_k - 1)^2} + \frac{q_k q_{k+1}}{(q_k q_{k+1} - 1)^2} \right) \in \mathbb{F}.$$

Then the first relation in Lemma 4.1 gives $\psi_k(q_k, q_{k+1}) \cdot \psi_\Lambda \;=\; \beta_k\, \psi_{s_k \cdot \Lambda}$. Rewriting this equality gives the explicit action of the element T_k on the basis element ψ_Λ as

$$T_k \cdot \psi_\Lambda \;=\; \beta_k\, \psi_{s_k \cdot \Lambda} \,-\, \left(\frac{\varepsilon}{q_k^{-1} q_{k+1} - 1} + \frac{\varepsilon}{q_k q_{k+1} - 1}\, C_k C_{k+1} \right) \psi_\Lambda\,. \tag{6.4}$$

iii) Finally, consider the remaining case where the tableau $s_k \cdot \Lambda$ is not standard. Then Proposition 6.1(c) gives $\psi_k(q_k, q_{k+1}) \cdot \psi_\Lambda \;=\; 0$. This equality can be rewritten as

$$T_k \cdot \psi_\Lambda \;=\; -\left(\frac{\varepsilon}{q_k^{-1} q_{k+1} - 1} + \frac{\varepsilon}{q_k q_{k+1} - 1}\, C_k C_{k+1} \right) \psi_\Lambda\,. \tag{6.5}$$

In every instance, the action of the generator T_k takes the element ψ_Λ to some element within the subspace V. This action can be extended to the full basis in V using the relations

(2.2). Thus left multiplication in the algebra $G'_n(q)$ by each generator T_k preserves the subspace V. Then $V_\lambda = G'_n(q)\cdot\psi_{\Lambda^r}\subseteq V$ and the part (a) of Theorem 6.2 is verified.

Consider the action of the affine generators $X_1,\dots,X_n$ on the subspace $V_\lambda\otimes_{\mathbb F}\mathbb K$ in M_{χ_0}. Let us fix the standard tableau $\Lambda\in\mathcal S_\lambda$. Given any $(x_1,\dots,x_n)\in\mathcal X$, we have the equality

$$\psi_\Lambda(x_1,\dots,x_n) \;=\; \pi_\chi(\Phi_{w_\Lambda})(1)$$

where χ is the generic character defined through (3.12) by $x_1,\dots,x_n$. At the special point $(q_1,\dots,q_n)$ Corollary 3.4 then gives

$$\pi_{\chi_0}(X_k)(\psi_\Lambda) \;=\; (w_\Lambda\cdot\chi_0)(X_k)\cdot\psi_\Lambda$$

for each $k=1,\dots,n$. But using the relations (3.2) we obtain

$$X_k\,C \;=\; C\,X_k^{-\nu_C(k)},\quad C\in\mathcal C.$$

Combining these results, for any index k the action of X_k on the vector $C\psi_\Lambda$ is given by

$$\pi_{\chi_0}(X_k)(C\psi_\Lambda) \;=\; (w_\Lambda\cdot\chi_0)(X_k^{-\nu_C(k)})\cdot C\psi_\Lambda\,;\quad C\in\mathcal C,\Lambda\in\mathcal S_\lambda\,.$$

The equality stated in Theorem 6.2(b) now follows from the definition of the character χ_0. In particular, the action of $X_1,\dots,X_n$ in M_{χ_0} preserves the subspace $V_\lambda\otimes_{\mathbb F}\mathbb K$. Since $\Lambda(1,1)=1$ for any standard tableau $\Lambda\in\mathcal S_\lambda$, we have $q_1=1$ and the result in (b) gives

$$\pi_{\chi_0}(X_1)(C\psi_\Lambda) \;=\; C\psi_\Lambda\,;\quad C\in\mathcal C,\ \Lambda\in\mathcal S_\lambda\,.$$

Thus the actions of X_1 and $\iota(X_1)=1$ in $V_\lambda\otimes_{\mathbb F}\mathbb K$ coincide. Theorem 6.2(c) now follows from Proposition 3.5. ■

The basis in V_λ described in Theorem 6.2(a) is an analogue of the Young basis [**3**] in an irreducible $H_n(q)$-module. Theorem 6.2(a) also shows that $\dim V_\lambda = 2^n\cdot m_\lambda$ where m_λ is the number of standard shifted tableaux with shape λ. The integer m_λ can be computed by an analogue of the hook formula [**11**, Example I.5.2] for the number n_ω of standard Young tableaux with shape ω. This analogue was first found by Morris [**12**, Theorem 2.1]; see also [**11**, Example III.7.8].

For the remainder of this paper, we will consider V_λ as a $G'_n(q)$-module. The vector space V_λ over $\mathbb F$ has a natural decomposition into the direct sum of the subspaces (6.2). For each $i=1,\dots,\ell_\lambda$ define a linear operator ρ_i in V_λ by

$$\rho_i\;:\;C\,\psi_\Lambda\;\mapsto\;C\,C_l\,\psi_\Lambda\,,\quad l=\Lambda(i,i)$$

where $\Lambda\in\mathcal S_\lambda$. Then we have the following easy observation; cf. [**13**, Theorem 8.3].

Proposition 6.3. *The operators $\rho_1,\dots,\rho_{\ell_\lambda}$ commute with the action of $G'_n(q)$ in V_λ.*

Proof. By definition the operators $\rho_1,\dots,\rho_{\ell_\lambda}$ commute with the action of $C_1,\dots,C_n\in G'_n(q)$ in V_λ. Now fix an index $i\in\{1,\dots,\ell_\lambda\}$ and a tableau $\Lambda\in\mathcal S_\lambda$. Put $l=\Lambda(i,i)$. It suffices to prove that in V_λ

$$T_k\cdot\rho_i(\psi_\Lambda)=T_k\,C_l\,\psi_\Lambda=\rho_i(T_k\,\psi_\Lambda)$$

for any $k\in\{1,\dots,n-1\}$. For $k\neq l,l-1$ the required second equality immediately follows from the third relation in (2.2) and from (6.3),(6.4),(6.5).

Suppose that the tableau $s_k\cdot\Lambda$ is standard and that $s_kw_\Lambda\succ w_\Lambda$. The cases when $s_k\cdot\Lambda$ is not standard or it is but $s_kw_\Lambda\prec w_\Lambda$ can be treated similarly, see the beginning of the proof

of Theorem 6.2. Let $l=k$, then $q_k=1$ by (4.6). Write $q_{k+1}=\delta$. Then by the first relation in (2.2) and by (6.3) we obtain the equalities

$$T_k\,C_k\,\psi_\Lambda=C_{k+1}\,T_k\,\psi_\Lambda=C_{k+1}\,\psi_{s_k\cdot\Lambda}-C_{k+1}\left(\frac{\varepsilon}{\delta-1}+\frac{\varepsilon}{\delta-1}\,C_kC_{k+1}\right)\psi_\Lambda$$
$$=C_{k+1}\,\psi_{s_k\cdot\Lambda}-\left(\frac{\varepsilon}{\delta-1}+\frac{\varepsilon}{\delta-1}\,C_kC_{k+1}\right)C_k\,\psi_\Lambda=\rho_i(T_k\,\psi_\Lambda)\,.$$

Now let $l=k+1$, then $q_{k+1}=1$. Write $q_k=\delta$. Then by the second relation in (2.2) and by (6.3) we obtain the equalities

$$T_k\,C_{k+1}\,\psi_\Lambda=C_k\,\psi_{s_k\cdot\Lambda}-C_k\left(\frac{\varepsilon}{\delta^{-1}-1}+\frac{\varepsilon}{\delta-1}\,C_kC_{k+1}\right)\psi_\Lambda+(\varepsilon\,C_{k+1}-\varepsilon\,C_k)\,\psi_\Lambda$$
$$=C_k\,\psi_{s_k\cdot\Lambda}\;-\;\left(\frac{\varepsilon}{\delta^{-1}-1}+\frac{\varepsilon}{\delta-1}\,C_kC_{k+1}\right)C_{k+1}\,\psi_\Lambda=\rho_i(T_k\,\psi_\Lambda)\,.\quad\blacksquare$$

Put $d_\lambda=0$ if the number ℓ_λ is even, and put $d_\lambda=1$ if ℓ_λ is odd. The assignment $C_i\mapsto\rho_i$ defines an action in V_λ of the Clifford algebra Z_λ over $\mathbb{C}$ with ℓ_λ generators. This algebra has a natural $\mathbb{Z}_2$-gradation: each of its generators $C_1,\dots,C_{\ell_\lambda}$ is odd. Take any minimal even idempotent in this algebra. For example, take any of the $2^{[\ell_\lambda/2]}$ pairwise orthogonal idempotents

$$2^{-[\ell_\lambda/2]}\cdot\big(1\pm C_1C_2\sqrt{-1}\,\big)\dots\big(1\pm C_{\ell_\lambda-d_\lambda-1}\,C_{\ell_\lambda-d_\lambda}\sqrt{-1}\,\big)\,.$$

The left ideal in Z_λ generated by this idempotent is irreducible under the left multiplication. It is absolutely irreducible if and only if the number ℓ_λ is even. Let γ_{Λ^r} be the image of this idempotent under the embedding of Z_λ into the Clifford algebra with n generators over $\mathbb{C}$ determined by

$$C_i\mapsto C_l\,,\quad l=\Lambda^r(i,i)\,.$$

Consider the left ideal U_λ in the algebra $G_n'(q)$ generated by the element $\gamma_{\Lambda^r}\,\psi_{\Lambda^r}$ as a $G_n'(q)$-module under the left multiplication. Then we get the following corollary to Proposition 6.3.

Corollary 6.4. *The $G_n'(q)$-module V_λ is a direct sum of $2^{[\ell_\lambda/2]}$ copies of the module U_λ.*

By the definition (4.6) we get $q_l=1$ if $l=\Lambda(i,i)$. Denote by γ'_{Λ^r} the image of the idempotent γ_{Λ^r} under the automorphism of the Clifford algebra with n generators over $\mathbb{C}$ determined by

$$C_l\;\mapsto\;C_{w_{\Lambda^r}^{-1}(l)}\;;\qquad l=1,\dots,n\,.$$

Then the next proposition follows immediately from Proposition 4.5.

Proposition 6.5. *We have the equality $\gamma_{\Lambda^r}\,\psi_{\Lambda^r}=\psi_{\Lambda^r}\,\gamma'_{\Lambda^r}$ in the algebra $G_n'(q)$.*

We now describe the *centre* $\mathrm{Z}(G_n(q))$ of the $\mathbb{Z}_2$-graded algebra $G_n(q)$. This is the collection of elements $Z\in G_n(q)$ such that the supercommutator $[\,Z\,,X\,]$ vanishes for all $X\in G_n(q)$. Note that the even component of $\mathrm{Z}(G_n(q))$ coincides with the even component of the centre in the usual sense. We use the Jucys-Murphy elements $J_1,\dots,J_n\in G_n(q)$ defined by (3.10).

Proposition 6.6. *a) The centre $\mathrm{Z}(G_n(q))$ of the $\mathbb{Z}_2$-graded algebra $G_n(q)$ consists precisely of the symmetric polynomials in the elements $J_1+J_1^{-1},\dots,J_n+J_n^{-1}$.*
b) The dimension of $\mathrm{Z}(G_n(q))$ over $\mathbb{C}(q)$ coincides with the number of strict partitions $\lambda\succ n$.

Proof. Propositions 3.2(b) and 3.5 imply that any symmetric polynomial in the elements $J_1 + J_1^{-1}, \ldots, J_n + J_n^{-1}$ is central (in the usual sense) for the algebra $G_n(q)$. Furthermore, these central elements are even since the Jucys-Murphy elements $J_1, \ldots, J_n$ are homogeneous with degree zero in the $\mathbb{Z}_2$-grading of $G_n(q)$. We have to prove that all these symmetric polynomials exhaust the centre $\mathrm{Z}(G_n(q))$, cf. [**9**, Section 3].

Let d denote the number of strict partitions of n. The algebra G_n can be obtained from the generic algebra $G_n(q)$ via the specialization $q \to 1$. The dimension of the centre does not decrease under this specialization. But the centre of the $\mathbb{Z}_2$-graded algebra G_n has dimension d [**18**, Lemma 6]. Thus the integer d provides an upper bound on the dimension of $\mathrm{Z}(G_n(q))$. It remains to show that the collection of symmetric polynomials in $J_1 + J_1^{-1}, \ldots, J_n + J_n^{-1}$ contains at least d linearly independent elements over $\mathbb{C}(q)$.

By the part (a) of Theorem 6.2 the elements $C\psi_\Lambda$ with $\Lambda \in \mathcal{S}_\lambda$ and $C \in \mathcal{C}$ form a $\mathbb{F}$-basis in V_λ. By the part (b) of the same theorem we have

$$(J_k + J_k^{-1}) \cdot C\psi_\Lambda \;=\; (q_k + q_k^{-1}) \cdot C\psi_\Lambda \;=\; \frac{2}{q+q^{-1}} \left(q^{2(j-i)+1} + q^{-2(j-i)-1} \right) \cdot C\psi_\Lambda$$

where $k = \Lambda(i,j)$; see (4.2) and (4.6). So any symmetric polynomial in $J_1 + J_1^{-1}, \ldots, J_n + J_n^{-1}$ acts in V_λ by a certain scalar from $\mathbb{C}(q)$. The collection of these scalars as we range over all symmetric polynomials, determines the partition λ uniquely. Indeed, the generating function $(\,t + J_1 + J_1^{-1}) \cdots (\,t + J_n + J_n^{-1})$ in t for the elementary symmetric polynomials has in V_λ the eigenvalue

$$2^n \cdot \prod_{k=1}^{n} \left(\frac{t}{2} + \frac{q^{2(j-i)+1} + q^{-2(j-i)-1}}{q+q^{-1}} \right).$$

It allows us to recover the unordered collection of the contents $j-i$ for the shifted Young diagram of λ; this collection determines the partition λ uniquely. Therefore the elementary symmetric polynomials in $J_1 + J_1^{-1}, \ldots, J_n + J_n^{-1}$ generate at least d linearly independent elements of $\mathrm{Z}(G_n(q))$. ■

Theorem 6.7. *The module U_λ over the $\mathbb{Z}_2$-graded algebra $G'_n(q)$*

a) is irreducible,

b) remains irreducible on passing to any extension of the field $\mathbb{F}$,

c) is absolutely irreducible if and only if the number ℓ_λ is even.

Proof. Let U^{-}_λ be a non-zero irreducible submodule in the $G^{-}_n(q)$-module $U_\lambda \otimes_{\mathbb{F}} \mathbb{K}$. In the proof of Proposition 6.6 we demonstrated that the central elements of the $\mathbb{Z}_2$-graded algebra $G_n(q)$ act in V_λ by certain scalars. These central elements act in U^{-}_λ by the same scalars. These scalars distinguish the modules U^{-}_λ for different strict partitions $\lambda \succ n$.

The $G'_n(q)$-module U_λ with $d_\lambda = 1$ cannot be absolutely irreducible by Proposition 6.3. But Theorem 6.2(a) and Corollary 6.4 along with the classical result of [**17**] show that

$$\sum_{\lambda \succ n} 2^{-d_\lambda} \cdot \left(\dim U_\lambda \right)^2 \;=\; \sum_{\lambda \succ n} \left(m_\lambda \cdot 2^{\,n-(\ell_\lambda/2)} \right)^2 \;=\; 2^n \cdot n!$$

which is exactly the dimension of the algebra $G'_n(q)$ over the field $\mathbb{F}$, see Proposition 2.1. So if U^{-}_λ is a proper submodule in $U_\lambda \otimes_{\mathbb{F}} \mathbb{K}$ or if the $G'_n(q)$-module U_λ with $d_\lambda = 0$ is not absolutely irreducible, we get a contradiction with Propositions 2.2 and 6.6(b). ■

In the course of the proof of Theorem 6.7 we also established the following fact.

Corollary 6.8. *The modules U_λ ranging over all strict partitions $\lambda \succ n$ form a complete set of irreducible pairwise non-equivalent $G'_n(q)$-modules.*

Thus we have demonstrated that $\mathbb{F}$ is a splitting field for the semisimple $\mathbb{C}(q)$-algebra $G_n(q)$.

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